A companion to Macdonald's book on $\mathfrak{p}$-adic spherical functions

Bill Casselman
University of British Columbia
cass@math.ubc.ca

April 7, 2005: 8:58 p.m.

When Ian Macdonald's book **Spherical functions on a group of p-adic type** first appeared, it was one of a very small number of publications concerned with representations of $\mathfrak{p}$-adic groups. At just about that time, however, the subject began to be widely recognized as indispensable in understanding automorphic forms, and the literature on the subject started to grow rapidly. Since it has by now grown so huge, in discussing here the subsequent history of some of Macdonald's themes I shall necessarily restrict myself only to things closely related to them. This will be no serious restriction since, as I hope to explain, some of the most interesting problems in all of representation theory are concerned with $\mathfrak{p}$-adic spherical functions. Along the way I'll reformulate from a few different perspectives what his book contains. I'll begin, in the next section, with a brief sketch of the main points, postponing most technical details until later.

Throughout, suppose k to be what I call a $\mathfrak{p}$-adic field, which is to say that it is either a finite extension of some $\mathbb{Q}_p$ or the field of Laurent polynomials in a single variable with coefficients in a finite field. Further let

- $\mathfrak{o}$ = the ring of integers of k;
- $\mathfrak{p}$ = the maximal ideal of $\mathfrak{o}$;
- ϖ = a generator of $\mathfrak{p}$;
- $q = |\mathfrak{o}/\mathfrak{p}|$, so that $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$.

Let $\mathbb{D}$ be a field of characteristic 0, which will play the role of coefficient field in representations. The minimal requirement on $\mathbb{D}$ is that it contain $\sqrt{q}$, but it will in the long run be convenient to assume that it is algebraically closed. It may usually be taken to be $\mathbb{C}$, but I want to emphasize that special properties of $\mathbb{C}$ are rarely required.

In writing this note I had one major decision to make about what class of groups I would work with. What made it difficult was that there were conflicting goals to take into account. On the one hand, I wanted to be able to explain a few basic ideas without technical complications. For this reason, I did not want to deal with arbitrary reductive groups, because even to state results precisely in this case would have required much distracting effort—effort, moreover, that would have just duplicated things explained very well in Macdonald's book. On the other, I wanted

to illustrate some of the complexities that Macdonald's book confronts. In the end, I chose to restrict myself to **unramified** groups. *I will suppose throughout this account that G is a reductive group defined over k arising by base extension from a smooth reductive scheme over $\mathfrak{o}$.* I hope that the arguments I present here are clear enough that generalization to arbitrary reductive groups will be straightforward once one understands their fine structure. I also hope that the way things go with this relatively simple class of groups will motivate the geometric treatment in Macdonald's book, which although extremely elegant is somewhat terse and short of examples. I'll say something later on in the section on root data about their structure.

Upon learning that I was going to be writing this essay, Ian Macdonald asked me to mention that Axiom V in Chapter 2 of his book is somewhat stronger than the corresponding axiom of Bruhat-Tits, and not valid for the type C–B_2 in their classification. Deligne pointed this out to him, and made the correction:

> Axiom V. *The commutator group $[U_\alpha, U_\beta]$ for α, $\beta > 0$ is contained in the group generated by the U_γ with $\gamma > 0$ and not parallel to α or β.*

Contents

Part I. The spherical function for unramified groups

1. Introduction

Let P be a minimal parabolic subgroup of G arising from a minimal parabolic subgroup of $G(\mathfrak{o})$. The group $G(\mathfrak{o})$ is a smooth scheme, and like its reduction modulo $\mathfrak{p}$ it is quasi-split, which is to say that P is a Borel subgroup. Let

- $N = N_P$, its unipotent radical;
- $M = M_P$, a copy in P of the Levi quotient P/N, which is a torus since P is a Borel subgroup;
- $\overline{P} =$ the parabolic subgroup **opposite** to P, with $P \cap \overline{P} = M$;
- $A = A_P$, the maximal split torus of M;
- $\mathcal{T} = M/M(\mathfrak{o})$, a free $\mathbb{Z}$-module containing the sub-lattice $\mathcal{A} = A/A(\mathfrak{o})$;
- $W = N_G(A)/M$, the Weyl group of the pair (G, A);
- $K = G(\mathfrak{o})$, a compact subgroup of G;
- $I =$ the **Iwahori subgroup** of K determined by P, the inverse image in K of the Borel subgroup of $G(\mathfrak{o}/\mathfrak{p})$ associated to P;
- $\delta_P = |\det \mathrm{Ad}_{\mathfrak{n}}|$, the modulus character of P;
- $\Sigma =$ the roots of the pair (G, A), the non-trivial eigencharacters of the adjoint action of A on the Lie algebra $\mathfrak{g}$;
- $\Sigma^+ =$ the subset of positive roots determined by P, so that

$$\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \ ;$$

- $\Delta =$ the basis of Σ^+, so that every root λ in Σ^+ is a sum of roots in Δ;
- $\Sigma^\vee =$ the **coroots** of the system, contained in the lattice

$$X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A) \ ;$$

- for each $\Theta \subset \Delta$

$$A_\Theta = \bigcap_{\alpha \in \Theta} \ker(\alpha) \ .$$

- Γ will be the Galois group of the maximal unramified extension k_{nr}/k, generated topologically by the Frobenius $\mathfrak{F}$

Thus $A_\emptyset = A$ itself and A_Δ is the maximal split torus in the centre of G. Representatives of W may always be chosen in K. The centralizer of A_Θ is the reductive component M_Θ of a unique parabolic subgroup P_Θ containing P. Let $\widetilde{M}_\Theta$ be the simply connected cover of its derived group. The only case we'll really care about is when Θ is a singleton $\{\alpha\}$.

A simple example that's good to keep in mind is that of $G = \mathrm{GL}_n(k)$, where P is the Borel subgroup of upper triangular matrices, $\overline{P}$ the lower triangular matrices,

W the symmetric group $\mathfrak{S}_n$, and $K = \mathrm{GL}_n(\mathfrak{o})$. Here each of the groups $\widetilde{M}_{\{\alpha\}}$ is a copy of SL_2.

I recall here some elementary facts about reductive groups over $\mathfrak{p}$-adic fields. Such a group G possesses as basis of open sets at 1 a sequence of compact open subgroups, which can be taken to be the **congruence subgroups**

$$G(\mathfrak{p}^n) = \{g \in G(\mathfrak{o}) \,|\, g \equiv I \pmod{\mathfrak{p}^n}\} \;.$$

Suppose $G(\mathfrak{o})$ to be embedded in $\mathrm{GL}_n(\mathfrak{o})$, defined by polynomial equations $P_i = 0$. Since
$$P(I + \varepsilon X) = P(I) + \varepsilon \langle dP_i(I), X \rangle + O(\varepsilon^2)$$
the Lie algebra of $G(\mathbb{F}_q)$ may be identified with the linear subspace of matrices X modulo $\mathfrak{p}$ such that all $\langle dP_i(I), X \rangle$ vanish. Since G is a smooth group scheme over $\mathfrak{o}$, it follows from Hensel's Lemma that whenever $n \geq 1$ the map taking X to $I + \varpi^n X$ induces an isomorphism of this Lie algebra with $G(\mathfrak{p}^n)/G(\mathfrak{p}^{n+1})$.

The space $C^\infty(G, \mathbb{D})$ of **smooth** functions on G with values in $\mathbb{D}$ will be made up of those that are locally constant.

Keep in mind that if S is any algebraic torus, then

$$X_*(S) = \mathrm{Hom}(\mathbb{G}_m, S), \quad X^*(S) = \mathrm{Hom}(S, \mathbb{G}_m)$$

are both free $\mathbb{Z}$-modules of finite rank, or what I'll call **lattices**, canonically dual to each other through the pairing into $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$—for all x in $k^\times$

$$\alpha(\beta^\vee(x)) = q^{\langle \alpha, \beta^\vee \rangle} \;.$$

I'll use additive notation for both lattices, and write x^λ for the image of x in $\mathbb{G}_m$ with respect to $\lambda^\vee$ in $X_*(S)$, and x^λ for the image of x in S with respect to λ in $X^*(S)$.

For a split torus S defined over a $\mathfrak{p}$-adic field ℓ something special occurs—the group $S(k)$ is isomorphic to $(k^\times)^n$ and the map taking $\chi^\vee$ to $\chi^\vee(\varpi)$ induces an isomorphism of $X_*(S)$ with $S(\ell)/S(\mathfrak{o}_\ell)$.

Another thing that makes the unramified groups simpler than arbitrary ones, as I have already mentioned, is that the Levi component M of the minimal parabolic subgroup is a torus. Yet another is this:

[lattices] 1.1. Proposition. *The embedding of A into M induces an isomorphism of the lattices $\mathcal{A} = A/A(\mathfrak{o})$ and $\mathcal{T} = M/M(\mathfrak{o})$.*

Proof. The torus M splits over the maximal unramified extension k_{nr} of k. Because A is the maximal split k-torus in M, inclusion identifies the lattices

$$X_*(A) = X_*(M)^{\mathcal{G}} \;.$$

Therefore

$$A(k)/A(\mathfrak{o}) \cong X_*(A)$$
$$\cong X_*(M)^{\mathcal{G}}$$
$$\cong \left(M(k_{\mathrm{nr}})/M(\mathfrak{o}_{\mathrm{nr}}) \right)^{\mathcal{G}} .$$

The short exact sequence of $\mathcal{G}$ modules

$$1 \longrightarrow M(\mathfrak{o}_{\mathrm{nr}}) \longrightarrow M(k_{\mathrm{nr}}) \longrightarrow M(k_{\mathrm{nr}})/M(\mathfrak{o}_{\mathrm{nr}}) \longrightarrow 1$$

leads to a long exact sequence

$$1 \longrightarrow M(\mathfrak{o}) \longrightarrow M(k) \longrightarrow \left(M(k_{\mathrm{nr}})/M(\mathfrak{o}_{\mathrm{nr}}) \right)^{\mathcal{G}}$$
$$\longrightarrow H^1\left(\mathcal{G}, M(\mathfrak{o}_{\mathrm{nr}})\right) \longrightarrow \ldots$$

Explicitly, the final map takes m in $M(k_{\mathrm{nr}})$ invariant modulo $M(\mathfrak{o})$ to $mm^{-\mathfrak{F}}$. So the Proposition follows from this:

[langs] **1.2. Lemma.** *If H is any unramified group over k then $H^1\left(\mathcal{G}, H(\mathfrak{o}_{\mathrm{nr}})\right) = 1$.*

Proof. Equivalently, it must be shown that the map $x \mapsto xx^{-\mathfrak{F}}$ is surjective on $H(\mathfrak{o}_{\mathrm{nr}})$. Filter $H(\mathfrak{o}_{\mathrm{nr}})$ by the kernels of the projections

$$H(\mathfrak{o}_{\mathrm{nr}}) \longrightarrow H(\mathfrak{o}_{\mathrm{nr}}/\mathfrak{p}_{\mathrm{nr}}^n)$$

which are surjective by Hensel's Lemma. Each graded term of this filtration is an algebraic group defined over the finite field $\mathfrak{o}/\mathfrak{p}$, and the classic result of Serge Lang ([Lang:1956]—see also Proposition 3, VI.4 of [Serre:1959] or [Müller:2003]—implies that $x \mapsto xx^{-\mathfrak{F}}$ is surjective on the group of its points rational over the algebraic closure of $\mathbb{F}_q$. An induction argument concludes. ◻

If χ is a character of M with values in $\mathbb{D}^\times$ that's trivial on $M(\mathfrak{o})$, it will induce one of $\mathcal{T}$. It is said to be an **unramified** character of M. It also induces a character of P trivial on N. The representation $\mathrm{Ind}(\chi \mid P, G)$ of G **induced** by χ is the right regular representation of G on the space of all smooth functions $f \colon G \to \mathbb{D}$ such that

$$f(pg) = \delta_P^{1/2}(p)\chi(p)f(g)$$

for all p in P, g in G. The factor $\delta_P^{1/2}$ is a useful normalization, as we'll see later. Since $G = PK$, this induced representation is an admissible representation of G. Its restriction to K is equal to $C^\infty(K \cap P\backslash K, \mathbb{D})$.

Its admissible dual is non-canonically isomorphic to $\mathrm{Ind}(\chi^{-1} \mid P, G)$. With a suitable choice of invariant measures, the duality sets

$$\langle f, \widetilde{f} \rangle = \int_K f(k)\widetilde{f}(k)\,dk .$$

These representations of G are called the **unramified principal series**.

[vK] **1.3. Proposition.** *The subspace of vectors in* $\mathrm{Ind}(\chi)$ *fixed by elements of K has dimension one.*

Proof. Again since $G = PK$. □

Conversely, if (π, V) is any irreducible admissible representation of G with $V^K \neq 0$, in which case π is said to be an **unramified representation** of G, then π embeds into some $\mathrm{Ind}(\chi)$. Let φ_χ be the unique function in $\mathrm{Ind}(\chi)$ fixed by K with $\varphi_\chi(1) = 1$.

Unramified representations are important for global arithmetical reasons. If $\widehat{\bigotimes} \pi_v$ is a representation of an adèlic group $G(\mathbb{A})$ then for all but a finite number of valuations v both the group $G(k_v)$ and the representation π_v will be unramified. That each unramified π_v embeds into an unramified principal series is where the application of representation theory to automorphic forms begins.

Assign G a Haar measure with $\mathrm{meas}(K) = 1$. The **Hecke algebra** $\mathcal{H}(G /\!/ K)$ is that of all left- and right-K-invariant functions of compact support on G, with convolution as its multiplication. If (π, V) is any admissible representation of G then every function in this Hecke algebra becomes an operator on V^K according to the formula

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg \ .$$

Since V^K is one-dimensional, they act as scalars. Hence:

[unramifiedchi] **1.4. Proposition.** *If χ is an unramified character of M and $\pi = \mathrm{Ind}(\chi)$, there exists a unique ring homomorphism c_χ from $\mathcal{H}(G /\!/ K)$ to $\mathbb{D}$ such that*

$$\pi(f)\varphi_\chi = c_\chi(f)\varphi_\chi$$

for every f in $\mathcal{H}(G /\!/ K)$.

These operators originated with the classical global Hecke operators $T(1, p)$ and $T(p, p)$ on automorphic forms on quotients of the upper half-plane by congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In the classical environment, the local nature of these operators was disguised by the fact that $\mathbb{Z}$ possesses strong unique factorization. This confusion extended at first to GL_n, but local operators were introduced when the methods of Tate's thesis were introduced to division algebras (avoiding until later the problems in analysis that arise with matrix algebras). The expositions in [Tamagawa:1963] and in §3.2 of [Shimura:1971] show some of this development.

For a given f the function $c_\chi(f)$ is in some sense a polynomial function of χ. The best way to see this is to look at the 'generic' or 'tautologous' unramified character χ of M, which takes m in M to its image modulo $M(\mathfrak{o})$ in the group algebra $R = \mathcal{H}(M /\!/ M(\mathfrak{o}))$. For example, if $G = \mathrm{GL}_n$ then $A \cong (k^\times)^n$ and R is isomorphic

to $\mathbb{D}[x_i^{\pm 1}]$. This tautologous character induces a representation of G on the space of locally constant functions on G with values in R such that

$$f(pg) = \delta_P^{1/2}(p)\chi(p)f(g)$$

for all p in P, g in G. The submodule of functions in this fixed by K is free of rank one over R, and we get therefore a ring homomorphism $\mathfrak{S} = c_\chi$ from $\mathcal{H}(G//K)$ to R. Any character χ of M with values in $\mathbb{D}^\times$ is a specialization of the generic one, and $c_\chi(f)$ then a specialization of $\mathfrak{S}(f)$.

If w is an element of the Weyl group W, then generically $\mathrm{Ind}(\chi)$ and $\mathrm{Ind}(w\chi)$ are isomorphic, which implies that for any f in $\mathcal{H}(G//K)$ the image of f in $\mathcal{H}(M//M(\mathfrak{o}))$ is W-invariant. Theorems going back to Hecke, Tamagawa, Satake, and Bruhat-Tits, with contributions from others, assert that this map, which is commonly called the **Satake transform** to acknowledge the nice axiomatic treatment in [Satake:1963], induces an isomorphism of $\mathcal{H}(G//K)$ with $\mathcal{H}(M//M(\mathfrak{o}))^W$. When $G = \mathrm{GL}_n$, for example, the ring $\mathcal{H}(M//M(\mathfrak{o}))$ is $\mathbb{D}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and tghe Weyl group is $\mathfrak{S}_n$, so every f is mapped onto a polynomial in the variables $x_i^{\pm 1}$ invariant under permutation of the variables.

Let A^{--} be the a in A such that $|\alpha(a)| \leq 1$ for all Δ, and let $\mathcal{T}^{--}$ be its image in $\mathcal{T}$. The cone $\mathcal{T}^{--}$ is a fundamental chamber for W in the lattice $\mathcal{T}$. The Cartan decomposition asserts that G is the disjoint union of cosets KtK where t ranges over $\mathcal{T}^{--}$. The Hecke algebra therefore has as linear basis the characteristic functions $\mathfrak{char}_{KtK}$, t in $\mathcal{T}^{--}$. When $G = \mathrm{GL}_n$ the quotient G/K can be interpreted as $\mathfrak{o}$-lattices in k^n, since K is the stabilizer of $\mathfrak{o}^n$, and this is just the principal divisor theorem.

The usual proof of the Satake isomorphism, which we'll see later on, is not constructive since it does not say which element of $\mathcal{H}(M//M(\mathfrak{o}))^W$ is the image of $\mathfrak{char}_{KtK}$. Although special cases had been known previously, it was apparently Langlands and Macdonald who independently came up with a general recipe. Langlands' version was announced in his 1967 lectures on Euler products (published first as notes in mimeographed form, later by the Yale Mathematics Department in [Langlands:1971]), in which he applied his result to deduce the convergence of certain L-functions. His reasoning was valid only for certain non-degenerate asymptotic values of t and only for Chevalley groups. Unitarity of cusp forms implied that matrix coefficients of local factor representations were bounded, and his asymptotic formula was good enough to allow him to deduce bounds on χ. The first version announced in [Macdonald:1968] was also valid only for Chevalley groups, but good for all t. His more precise result was more difficult to prove, but also more elegant. In his book, Macdonald exhibits a recipe for arbitrary simply connected groups, applying results of Bruhat-Tits on the fine structure of G. Extending Macdonald's formula to all reductive $\mathfrak{p}$-adic groups was apparently done first in [Casselman:1980].

The first step in explaining Macdonald's result is a slight reformulation of the problem. If $f = \mathfrak{char}_{KtK}$ we can write

$$
\begin{aligned}
\pi(f)\varphi_\chi &= \int_G f(g)R(g)\varphi_\chi \, dg \\
&= \sum_{x \in KtK/K} R_x \varphi_\chi \\
&= \sum_{k \in K/K \cap tKt^{-1}} R_{kt}\varphi_\chi
\end{aligned}
$$

and since $c_\chi(f)$ is $\pi(f)\varphi_\chi$ evaluated at 1, this leads to

$$
\begin{aligned}
c_\chi(f) &= |KtK/K| \int_K \varphi_\chi(kt) \, dk \\
&= |KtK/K| \int_K \varphi_\chi(kt)\varphi_{\chi^{-1}}(k) \, dk \\
&= |KtK/K| \, \langle R_t \varphi_\chi, \varphi_{\chi^{-1}} \rangle \, .
\end{aligned}
$$

In other words

$$
c_\chi(f) = \frac{\mathrm{meas}(KtK)}{\mathrm{meas}(K)} \, \Phi_\chi(t)
$$

where $\Phi_\chi(g)$ is the **spherical function** associated to $\mathrm{Ind}(\chi)$, that is to say the matrix coefficient

$$
\langle R_g \varphi_\chi, \varphi_{\chi^{-1}} \rangle = \int_K \varphi_\chi(kg) \, dk \, .
$$

One merit of working with Φ_χ, as we'll see in a moment, is that it has a more uniform expression than $c_\chi(f)$. Another is that it introduces the more general question of how to evaluate matrix coefficients explicitly. But before I exhibit that expression, I'll first say something about the volume factor.

We have put on G a Haar measure with $\mathrm{meas}(K) = 1$, but there is another equally natural way to choose one. According to what is often called the **Bruhat decomposition**, the group G is covered by disjoint double cosets $PwP = PwN$ as w ranges over W, and $Pw_\ell N$ is the unique one of these that's open, where w_ℓ is the longest element of W. If we put on P and N the Haar measures with $P(\mathfrak{o})$ and $N(\mathfrak{o})$ of measures 1, then there is a unique Haar measure on G such that the integral of f over G is

$$
\int_P d_\ell p \int_N f(pw_\ell n) \, dn
$$

whenever f has support on $Pw_\ell N$. These two Haar measures on G must be proportional to each other—there exists a rational constant μ_G such that

$$
\int_G f(g) \, dg = \mu_G \int_P dp \int_N f(pw_\ell n) \, dn
$$

where the integral on the left is with respect to that with $\mathrm{meas}(K) = 1$. It is not difficult to evaluate this constant explicitly. We start with the following, which will be useful later on.

[langs]1.5. Lemma. (Iwahori factorization) *We can factor the Iwahori subgroup as*

$$I = N(\mathfrak{o})M(\mathfrak{o})\overline{N}(\mathfrak{p})$$

where $\overline{N}(\mathfrak{p})$ is the subgroup of elements of $\overline{N}(\mathfrak{o})$ reducing to 1 modulo $\mathfrak{p}$.

Proof. This can be found as 2.6.4 in Macdonald's book, but it's easy enough to prove directly for unramified groups. If g lies in I, then modulo $\mathfrak{p}$ it lies in $P(\mathbb{F}_q)$, and because $P(\mathfrak{o})$ is smooth there exists g in $P(\mathfrak{o})$ with the same image (Hensel's Lemma). From now on we work by induction to prove that g has the factorization we want. This means we might as well assume g to be in $N(\mathfrak{o})M(\mathfrak{o})\overline{N}(\mathfrak{p})$ modulo $\mathfrak{p}^n$, and try to factor it similarly modulo $\mathfrak{p}^{n+1}$. But this follows from the conclusion of the previous paragraph, and the decomposition

$$\mathfrak{g}_{\mathbb{F}_q} = \overline{\mathfrak{n}}_{\mathbb{F}_q} + \mathfrak{m}_{\mathbb{F}_q} + \mathfrak{n}_{\mathbb{F}_q} \,.\ \square$$

As a consequence of the lemma, the coset $Iw_\ell I$ is completely contained in $Pw_\ell N$, so to compare the two measures we can integrate over it. This gives us

$$\mu_G = \frac{\mathrm{meas}(Iw_\ell I)}{\mathrm{meas}(K)} \,.$$

The Bruhat decomposition for $G(\mathbb{F}_q)$ tells us that K is the disjoint union of the cosets IwI as w ranges over W. For each w in W let $q(w) = |IwI/I|$. Thus

$$\mu_G = \frac{q(w_\ell)}{\sum_w q(w)} \,.$$

If $\ell(xy) = \ell(x) + \ell(y)$ then $q(xy) = q(x)q(y)$. Hence $q(ww_\ell)q(w) = q(w_\ell)$ and

$$\mu_G = \frac{1}{\sum_W q(w)q(w_\ell)^{-1}} = \frac{1}{\sum_W q(ww_\ell)^{-1}} = \frac{1}{\sum_W q(w)^{-1}} \,.$$

It is not difficult to compute individual $q(w)$, since if w has a reduced expression $w = s_1 \ldots s_n$ then $q(w) = q(s_1) \ldots q(s_n)$. If G is split, then $q(s) = q$ for all s in S, and $q(w) = q^{\ell(w)}$. If $G = \mathrm{GL}_n$, the quotient K/I may be identified with a flag manifold over $\mathfrak{o}/\mathfrak{p}$, and has size

$$(1 + q)(1 + q + q^2) \ldots (1 + q + q^2 + \cdots + q^{n-1})$$

giving

$$\mu_G = \frac{1}{(1 + q^{-1})(1 + q^{-1} + q^{-2}) \ldots (1 + q^{-1} + q^{-2} + \cdots + q^{-(n-1)})} \,.$$

The analogous problem arises with real reductive groups as well, and was first dealt with by Harish-Chandra. A formula for μ_G arises naturally in the context of Tamagawa measures, for example in Langlands' paper [Langlands:1966] at the Boulder conference. In view of the occurrence of the constant so often in Macdonald's book, it is of interest that he wrote later the short paper [Macdonald:1980] on the analogous question for certain real groups.

This same constant occurs in many contexts, not always for transparent reasons. So it is here. Suppose t to be an element of $\mathcal{T}$, and let a be an element of A equivalent to it modulo $M(\mathfrak{o})$. The centralizer of a is a reductive group M_t, and if t lies in $\mathcal{T}^{--}$ this is the Levi component of a parabolic subgroup containing P. For $t = 1$ this will be G itself, and for a generic t it will just be $\mathcal{T}$. An equivalent way to classify elements of $\mathcal{T}$ is by the subgroup of W fixing them, which will be the Weyl group of the group M_t. At any rate, we can partition $\mathcal{T}^{--}$ into subsets $\mathcal{T}_M^{--}$ in this way, essentially by the walls of the Weyl chamber $\mathcal{T}^{--}$. The formula for the volume of KtK depends on which of these sets t belongs to.

[volume] 1.6. Proposition. *For all t in $\mathcal{T}_M^{--}$*

$$|KtK/K| = \left(\frac{\mu_M}{\mu_G} \right) \delta_P^{-1}(t) .$$

This is proven in §3.2 of Macdonald's book for simply connected groups G, but the same proof is valid in general. The basic idea is to decompose KtK into double cosets IxI where now x runs through certain elements of an affine Weyl group we'll meet later on.

A simple example will make the nature of this result clearer, perhaps. Let $G = GL_2$ and take

$$t = \begin{bmatrix} \varpi^r & 0 \\ 0 & 1 \end{bmatrix}$$

with $r > 0$. Then

$$|KtK/K| = |K/K \cap tKt^{-1}| .$$

Now $K \cap tKt^{-1}$ is the group of matrices

$$\begin{bmatrix} a_{1,1} & \varpi^r a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

with all $a_{i,j}$ integral. The quotient $K/K \cap tKt^{-1}$ therefore factors over $\mathbb{P}^1(\mathbb{F}_q)$ with a fibre of cardinality q^{k-1}, so that in this case

$$|KtK/K| = q^{r-1}(q+1) = (1 + q^{-1}) q^r ,$$

which agrees with the Proposition. It is instructive to do a similar calculation for various t in GL_3.

For each root α, recall that $A_{\{\alpha\}}$ is the subgroup of a in A with $\alpha(a) = 1$, $M_{\{\alpha\}}$ its centralizer, a reductive group of semi-simple split rank one, and $\widetilde{M}_{\{\alpha\}}$ the simply connected covering of the derived group of $M_{\{\alpha\}}$. The group $\widetilde{M}_{\{\alpha\}}$ is an unramified, simply connected, simple group of split rank one. It is not too difficult to see that there are very few such groups. Indeed, any of them must be isomorphic to one of two kinds:

- $\mathrm{SL}_2(F)$ where F is an unramified finite extension of k;
- an unramified unitary group $\mathrm{SU}_3(E/F)$ of dimension 3 defined over an unramified finite extension F.

Any group over F, of course, determines one over k by restriction of scalars.

In the second case, let E/F be the unramified quadratic extension of F, and let H be the Hermitian form $x_1\overline{x}_3 + x_2\overline{x}_2 + x_3\overline{x}_1$ on F^3, where $\overline{x}$ is the quadratic conjugate of x in E. The group I call $\mathrm{SU}_3(E/F)$ is the special unitary group of H, that of all matrices X in $\mathrm{GL}_3(E)$ of determinant 1 such that

$$ {}^tX J \overline{X} = J \ \text{ or } J \, {}^tX^{-1} J = \overline{X} $$

where

$$ J = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} . $$

The map $X \mapsto J \, {}^tX^{-1} J$ is an involutory automorphism of SL_3 taking the group of upper triangular matrices to itself. If

$$ X = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} $$

then

$$ J \, {}^tX^{-1} J = \begin{bmatrix} c^{-1} & & \\ & b^{-1} & \\ & & a^{-1} \end{bmatrix} $$

so the diagonal matrices in SU_3 are the

$$ \begin{bmatrix} x & & \\ & \overline{x}/x & \\ & & \overline{x}^{-1} \end{bmatrix} . $$

If

$$ X = \begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix} $$

then

$$J \ {}^t X^{-1} J = \begin{bmatrix} 1 & z & xz - y \\ & 1 & x \\ & & 1 \end{bmatrix}$$

so the unipotent ones are the

$$\begin{bmatrix} 1 & x & y \\ & 1 & \overline{x} \\ & & 1 \end{bmatrix}$$

with $y + \overline{y} = x\overline{x}$.

In either case, let α be the unique indivisible positive root, and let $a_\alpha = \alpha^\vee(\varpi)$. It is the image of an element $\widetilde{a}_\alpha$ in $\widetilde{M}_\alpha$, namely

$$\widetilde{a}_\alpha = \begin{cases} \begin{bmatrix} \varpi & \\ & \varpi^{-1} \end{bmatrix} & \text{if } \widetilde{M}_\alpha = SL_2(E) \\[2em] \begin{bmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{bmatrix} & \text{if } \widetilde{M}_\alpha = SU_3(E) \end{cases}$$

For each α in Σ define

$$c_{\alpha,\chi} = \begin{cases} \dfrac{1 - q_E^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)} & \text{if } \widetilde{M}_\alpha = SL_2(E) \\[1.5em] \dfrac{\left(1 - q_E^{-2}\chi(a_\alpha)\right)\left(1 + q_E^{-1}\chi(a_\alpha)\right)}{1 - \chi^2(a_\alpha)} & \text{if } \widetilde{M}_\alpha = SU_3(E) \end{cases}$$

and then set

$$\gamma_\chi = \prod_{\alpha > 0} c_{\alpha,\chi^{-1}} \ .$$

If w is in W and χ is a character of M, then $w\chi$ is the character defined by

$$w\chi(m) = \chi(w^{-1}mw) \ .$$

A **regular** or **non-singular** character is one not fixed by any w in W. Macdonald's formula for Φ_χ asserts that

[regular] **1.7. Proposition.** *If χ is a regular unramified character of M and t in $\mathcal{T}^{--}$ then*

$$\Phi_\chi(t) = \mu_G \ \delta_P^{1/2}(t) \sum_{w \in W} \gamma_{w\chi} \ w\chi(t) \ .$$

There is also a version for singular characters, obtained by taking the limit of the one above. This formula is in some sense the analogue for $\mathfrak{p}$-adic groups of the Weyl

character formula for real reductive groups, and indeed there is a subtle relationship between the two.

The constant μ_G is, as we have seen, a ratio of measures on G. The proof of Macdonald's formula to be given later will make this role clear.

The proof of the formula for Φ_χ given in Macdonald's book has many admirable features, but I think it is fair to say that subsequent results on admissible representations of reductive $\mathfrak{p}$-adic groups allow one to understand it better. The proof I presented in my 1980 paper had the virtue that it extended without trouble to generic Hecke algebras, in which q becomes simply a variable and the group has disappeared. This is shown particularly in [Kato:1982], which used also results from [Matsumoto:1977]. It has become more or less the standard proof; it is the one presented, for example, in [Haines-Kottwitz-Prasad:2004]. In retrospect, however, I find this proof unsatisfactory. The trouble with it, as with all the proofs that work with a generic variable q, is that it doesn't distinguish between quite different groups where the value of q is the same. That is to say, $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{F}_p)$ are equivalent as far as this technique is concerned. Of course this is a virtue in many ways, but it doesn't tell you how to deal with ramified representations. I'd like to think that the argument I sketch later on in this account is in several ways preferable. It is based on an idea that first occurs in Langlands' original exposition, and its relevance to the present question is more explicitly formulated in [Waldspurger:1989]. The point of this argument will be to place the formula in a larger context, so that one understands to what extent such a result holds for arbitrary matrix coefficients on G, and to what extent the spherical function itself is special.

Example. Let $G = \mathrm{SL}_3(k)$ and $\chi = \delta_P^{-1/2}$. The spherical function itself in this case is identically 1. What expression does the formula produce? There are three coroots α and their coroot images a_α are

$$
\begin{bmatrix} \varpi & & \\ & \varpi^{-1} & \\ & & 1 \end{bmatrix}, \quad
\begin{bmatrix} 1 & & \\ & \varpi & \\ & & \varpi^{-1} \end{bmatrix}, \quad
\begin{bmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{bmatrix}
$$

mapped by $\delta_P^{1/2}$ to q, q, and q^2. So the formula tells us that

$$
\frac{1}{1 + 2q^{-1} + 2q^{-2} + q^3} \frac{(1 - q^{-2})(1 - q^{-2})(1 - q^{-3})}{(1 - q^{-1})(1 - q^{-1})(1 - q^{-2})}
$$
$$
= \frac{(1 + q^{-1})(1 + q^{-1} + q^{-2})}{1 + 2q^{-1} + 2q^{-2} + q^3} = 1 \; .
$$

For every simply connected split group G, say of rank n, one obtains a similar result when χ is $\delta_P^{-1/2}$. The well known result of [Kostant:1956] implies that

$$
\prod_{\alpha > 0} \frac{1 - q^{-1}\delta_P^{-1/2}(a_\alpha)}{1 - \delta_P^{-1/2}(a_\alpha)} = \prod_{i=1}^{n} \frac{1 - q^{-m_i - 1}}{1 - q^{-1}}
$$

where the m_i are the Coxeter exponents of W. That this in turn is equal to $\sum_{w \in W} q^{-\ell(w)}$ is proven in [Solomon:1966]. This is also related to the computation of the number of points on the flag manifold in terms of its cohomology and to the calculation of Tamagawa numbers in [Langlands:1966].

Example. Let $G = \mathrm{PGL}_2(k)$. Let

$$\omega = \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix} \text{ modulo the centre}, \quad X = \chi(\omega) .$$

Here $a_\alpha = \omega^2$ and

$$\delta_P^{1/2}(\omega) = q^{-1/2} .$$

Thus ω generates $A^{--}/A(\mathfrak{o})$, and Macdonald's formula tells us that

$$\begin{aligned}
\Phi_\chi(\omega^m) &= \frac{\delta_P^{1/2}(\omega^m)}{1 + q^{-1}} \left[\frac{1 - q^{-1}\chi^{-1}(a_\alpha)}{1 - \chi^{-1}(a_\alpha)} \chi(\omega^m) + \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)} \chi^{-1}(\omega^m) \right] \\
&= \frac{q^{-m/2}}{1 + q^{-1}} \left[\frac{1 - q^{-1}X^{-2}}{1 - X^{-2}} X^m + \frac{1 - q^{-1}X^2}{1 - X^2} X^{-m} \right] \\
&= \frac{q^{-m/2}}{1 + q^{-1}} \left[\left(\frac{X^{m+1} - X^{-(m+1)}}{X - X^{-1}} \right) - q^{-1} \left(\frac{X^{m-1} - X^{-(m-1)}}{X - X^{-1}} \right) \right]
\end{aligned}$$

This leads to a formula for $c_\chi(f)$, the image of f under the Satake homomorphism. If $m = 0$ then

$$c_\chi(\mathfrak{char}_{K\omega^m K}) = c_\chi(\mathfrak{char}_K) = 1$$

and otherwise

$$c_\chi(\mathfrak{char}_{K\omega^m K}) = q^{m/2} \left[\left(\frac{X^{m+1} - X^{-(m+1)}}{X - X^{-1}} \right) - q^{-1} \left(\frac{X^{m-1} - X^{-(m-1)}}{X - X^{-1}} \right) \right] .$$

Following an earlier suggestion, let χ be the tautologous character, with every element of $\mathcal{T}$ mapping to its image in $\mathcal{T}$. In this case X is just the image of ω in $\mathcal{T}$. It is legitimate then to write the formula above as

$$\begin{aligned}
\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) &= q^{m/2} \left[\left(\frac{\omega^{m+1} - \omega^{-(m+1)}}{\omega - \omega^{-1}} \right) - q^{-1} \left(\frac{\omega^{m-1} - \omega^{-(m-1)}}{\omega - \omega^{-1}} \right) \right] \\
&= q^{m/2} \left(\omega^m + \omega^{m-2} + \cdots + \omega^{-m} \right) - q^{m-2/2} \left(\omega^{m-1} + \omega^{m-3} + \cdots + \omega^{-(m-1)} \right)
\end{aligned}$$

which makes the connection with $\mathcal{H}(M//M(\mathfrak{o}))$ inescapable. In any event, this is a fascinating expression if you are addicted to q-ology, and we'll look at it again later on to see if we can understand what it's trying to tell us.

Example. Let $G = \mathrm{SL}_2$,

$$\omega = \begin{bmatrix} \varpi & \\ & \varpi^{-1} \end{bmatrix} .$$

Here $a_\alpha = \omega$ and

$$\delta_P^{1/2}(\omega) = q^{-1} \ .$$

For $m > 0$

$$\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) = q^m \left[\left(\frac{1 - q^{-1}\omega^{-1}}{1 - \omega^{-1}} \right) \omega^m + \left(\frac{1 - q^{-1}\omega}{1 - \omega} \right) \omega^{-m} \right]$$

$$= q^m \left(\omega^m + \omega^{m-1} + \cdots + \omega^{-m} \right) - q^{m-1} \left(\omega^{m-1} + \omega^{m-2} + \cdots + \omega^{-(m-1)} \right) \ .$$

Example. Let $G = SU_3$,

$$\omega = \begin{bmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{bmatrix} \ .$$

Here $\omega = a_\alpha$ and

$$\delta_P^{1/2}(\omega) = q^{-2} \ .$$

For $m > 0$

$$\mathfrak{S}(\mathfrak{char}_{K\omega^m K})$$

$$= q^{2m} \left[\frac{(1 - q^{-2}\omega^{-1})(1 + q^{-1}\omega^{-1})}{1 - \omega^{-2}} \, \omega^m + \frac{(1 - q^{-2}\omega)(1 + q^{-1}\omega)}{1 - \omega^2} \, \omega^{-m} \right]$$

$$= q^{2m} \left[\frac{(\omega - q^{-2})(1 + q^{-1}\omega^{-1})}{\omega - \omega^{-1}} \, \omega^m - \frac{(\omega^{-1} - q^{-2})(1 + q^{-1}\omega)}{\omega - \omega^{-1}} \, \omega^{-m} \right]$$

$$= q^{2m} \left[\frac{\omega^{m+1} - \omega^{-(m+1)}}{\omega - \omega^{-1}} \right] - q^{2m-2} \left[\frac{\omega^m - \omega^{-m}}{\omega - \omega^{-1}} \right]$$

$$+ q^{2m-1} \left[\frac{\omega^m - \omega^{-m}}{\omega - \omega^{-1}} \right] - q^{2m-3} \left[\frac{\omega^{m-1} - \omega^{-(m-1)}}{\omega - \omega^{-1}} \right] \ .$$

This formula is a bit more puzzling than the ones for PGL_2 and SL_2.

In the next few sections I'll fill in some details of things I have only sketched here. The principal aim of these sections will be to sketch proofs of the principal results concerning unramified representations:

[satakeiso] 1.8. Theorem. (Satake isomorphism) *If χ takes t to its image modulo $M(\mathfrak{o})$, then the map taking f in the Hecke algebra $\mathcal{H}(G//K)$ to $c_\chi(f)$ induces an isomorphism of this Hecke algebra with $\mathcal{H}(M//M(\mathfrak{o}))^W$.*

[embeds] 1.9. Theorem. *Any unramified admissible representation (π, V) of G embeds into some $\mathrm{Ind}(\chi)$.*

The character χ will generally not be unique, but its W-orbit will be. In any case, the subspace V^K will consequently have dimension 1, and there exists a ring homomorphism c_π from $\mathcal{H}(G//K)$ to $\mathbb{D}$ such that

$$\pi(f)v = c_\pi(f)v$$

for all f in $\mathcal{H}(G//K)$ and v in V^K. I call this the **characteristic homomorphism** of π.

[heckemap] **1.10. Theorem.** *If c is any ring homomorphism from $\mathcal{H}(G//K)$ to $\mathbb{D}$, there exists up to isomorphism exactly one irreducible admissible representation (π_c, V) of G whose characteristic homomorphism is c.*

[macformula] **1.11. Theorem.** *Macdonald's formula for the spherical function:*

$$\Phi_\chi(t) = \mu_G \; \delta_P^{1/2}(t) \sum_{w \in W} \gamma_{w\chi} \; w\chi(t) \; .$$

This discussion will include a very brief introduction to admissible representations. After these things have been dealt with in this Part I., I'll take up even more briefly in Part II. some questions discussed in Macdonald's book as well as a few more recent developments concerned with unramified representations that were only dimly foreshadowed there. Some of these questions are:

- *What is the explicit inverse of the Satake transform?*
- *Which unramified representations are unitary?*
- *Which unramified representations are expected to occur globally on arithmetic quotients?*
- *What are the characters of unramified representations?*
- *Are unramified representations for different groups related?*
- *How do things change if G is ramified?*

2. The unramified principal series

There has been one extremely simple but far-reaching change since Macdonald's book first appeared in how representations of $\mathfrak{p}$-adic groups are dealt with. Real analysis (should I say instead 'authentic' analysis) has largely disappeared from the subject, to be replaced by algebra. Recall that $\mathbb{D}$ is an arbitrary field of characteristic 0 containing $\sqrt{q}$. An **admissible representation** π of G over the field $\mathbb{D}$ is a representation of G on a $\mathbb{D}$-vector space V with these two properties:

- (Smoothness) The isotropy group of every v in V is open in G;
- (Admissibility) if K is any compact open subgroup of G, then the subspace of vectors in V fixed by K has finite dimension.

This elementary innovation was introduced rather casually in the book [Jacquet-Langlands:1971] and simplified the subject enormously.

If (π, V) is an admissible representation of G then the space of smooth $\mathbb{D}$-linear functions on V makes up its **admissible dual** $(\widetilde{\pi}, \widetilde{V})$, with G acting contragrediently:

$$\langle v, \widetilde{\pi}(g)\widetilde{v} \rangle = \langle \pi(g^{-1})v, \widetilde{v} \rangle \ .$$

The original representation is itself the admissible dual of $\widetilde{V}$. Asssociated to π is the $\mathbb{D}$-module of its **matrix coefficients**, the smooth functions in $C^\infty(G, \mathbb{D})$ of the form $\langle \pi(g)v, \widetilde{v} \rangle$ for v in V, $\widetilde{v}$ in $\widetilde{V}$.

The simplest way to construct admissible representations is by **parabolic induction**. Here, we'll look only at induction from the minimal parabolic subgroup P. If

$$\chi \colon M \to \mathbb{D}^\times$$

is a smooth multiplicative character of M, it induces a character of P trivial on N. The representation of G induced by χ from P is the right regular representation of G on

$$\mathrm{Ind}(\chi \mid P, G)$$
$$= \{ f \in C^\infty(G, \mathbb{D}) \mid f(pg) = \delta_P^{1/2}(p)\chi(p)f(g) \text{ for all } p \in P, g \in G \} \ .$$

This is an admissible representation of G since $P\backslash G$ is compact.

If $\chi = \delta_P^{-1/2}$ then $\mathrm{Ind}(\chi)$ is just $C^\infty(P\backslash G)$. If $\chi = \delta_P^{1/2}$ then $\mathrm{Ind}(\chi)$ will be the space of **smooth one-densities** on $P\backslash G$. Let me explain.

The group G is unimodular, but the parabolic subgroup P is not. This implies that there is no G-invariant measure on the quotient $P\backslash G$. Instead, we have the following situation. Fix a Haar measure dg on G such that $\mathrm{meas}(K) = 1$, and fix a

left-invariant Haar measure $d_\ell p$ on P such that $\mathrm{meas}(P \cap K) = 1$. Because $G = PK$ and G and K are both unimodular

$$\int_G f(g)\,dg = \int_P d_\ell p \int_K f(pk)\,dk \; .$$

The function

$$\overline{f}(x) = \int_P f(px)\,d_\ell p$$

satisfies

$$\begin{aligned}
\overline{f}(p_0 x) &= \int_P f(pp_0 x)\,d_\ell p \\
&= \int_P f(px)\,d_\ell p p_0^{-1} \\
&= \delta_P(p_0) \int_P f(px)\,d_\ell p \\
&= \delta_P(p_0)\overline{f}(x) \; ,
\end{aligned}$$

or in other words lies in $\mathrm{Ind}(\delta_P^{1/2})$. The integral formula just above therefore implies that integration

$$\int_K \overline{f}(k)\,dk$$

defines on $\mathrm{Ind}(\delta_P^{1/2})$ a G-invariant functional, which I'll write as integration

$$\int_{P\backslash G} \overline{f}(x)\,dx \; .$$

The pairing

$$\langle f, \overline{f}\rangle_{P\backslash G} = \int_{P\backslash G} f(x)\overline{f}(x)\,dx$$

identifies $\mathrm{Ind}(\delta_P^{1/2})$ with the smooth dual of $\mathrm{Ind}(\delta_P^{-1/2}) = C^\infty(P\backslash G)$, which is why it is called the space of one-densities on $P\backslash G$.

For explicit calculation it is useful to know that

$$\int_P f(p)\,d_\ell p = \int_M \delta_P^{-1}(m)\,dm \int_N f(nm)\,dn \; .$$

If f lies in $\mathrm{Ind}(\chi)$ and $\widetilde{f}$ in $\mathrm{Ind}(\chi^{-1})$ then the product $f \cdot \widetilde{f}$ lies in $\mathrm{Ind}\big(\delta_P^{1/2}\big)$, and the pairing that can be formally expressed as

$$\langle f, \widetilde{f}\rangle = \int_{P\backslash G} f(x)\widetilde{f}(x)\,dx$$

defines an isomorphism of $\mathrm{Ind}(\chi^{-1})$ with the admissible dual of $\mathrm{Ind}(\chi)$—this simple formulation of duality is the one of the principal reasons for normalization by the factor $\delta_P^{-1/2}$. (There are others we shall see later.) If $\mathbb{D} = \mathbb{C}$ and χ is unitary, then $\chi^{-1} = \overline{\chi}$ and $\mathrm{Ind}(\chi)$ is therefore a unitary representation of G.

If χ is trivial on $K \cap P$ or equivalently on $M \cap K = M(\mathfrak{o})$ it is called **unramified**. Since $M/M(\mathfrak{o}) \cong \mathbb{Z}^n$, the unramified characters of M are in bijection with a product of n copies of $\mathbb{D}^\times$. These are the $\mathbb{D}$-rational points on a split torus defined over $\mathbb{D}$. This peculiar fact is not an unimportant accident, as we shall see.

When χ is unramified, since $G = PK$ the subspace of elements fixed by K has dimension one over $\mathbb{D}$. Let φ_χ be the unique element of this module with $\varphi_\chi(1) = 1$.

3. The Iwahori Hecke algebra

There are two particular Hecke algebras of importance to the theory of spherical functions. The first is $\mathcal{H}(G, G(\mathfrak{o}))$—the one the Satake isomorphism is concerned with—and the second is the **Iwahori Hecke algebra** $\mathcal{H}(G, I)$. It is the second that we'll be concerned with in this section. Macdonald explains its structure when G is simply connected, and the general case derives from that.

Let $\mathcal{W}$ be the quotient $N_G(A)/M(\mathfrak{o})$. It fits into an exact sequence

$$1 \longrightarrow T \longrightarrow \mathcal{W} \longrightarrow W \longrightarrow 1$$

since $W = N_G(A)/M$ and $T = M/M(\mathfrak{o})$. This sequence splits, since the intersection $K \cap N_G(A)$ contains representatives of W and $K \cap N_G(A)/K \cap M(\mathfrak{o})$ projects isomorphically onto W. But this is not the only interesting description of $\mathcal{W}$.

Let $\widetilde{G}$ be the simply connected cover of the derived group of G. Its image in G is normal, and contains W. Let $\widetilde{A}$ be the inverse image in $\widetilde{G}$ of A, $\widetilde{T}$ the image of that in T. The image of $X_*(\widetilde{A})$ in $X_*(A)$ is spanned by the coroots $\alpha^\vee$, so $\widetilde{T}$ is spanned by the $a_\alpha = \alpha^\vee(\varpi)$. Let Ω be the quotient of T by $\widetilde{T}$, which may be identified with the quotient of $\mathcal{W}$ by the subgroup $\widetilde{W}$ generated by $\widetilde{T}$ and W, which is normal in $\mathcal{W}$:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{T} & \longrightarrow & T & \longrightarrow & \Omega & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \widetilde{W} & \longrightarrow & \mathcal{W} & \longrightarrow & \Omega & \longrightarrow & 1
\end{array}
$$

As Macdonald explains, the group $\widetilde{W}$ is an affine Weyl group. Let me recall here what that means. The group $X_*(A)$ is a lattice in the real vector space $X_*(A) \otimes \mathbb{R}$. The roots in Σ are linear functions in the dual lattice $X^*(A)$. To each α in Σ is associated a coroot $\alpha^\vee$ in $X_*(A)$, and reflection s_α in the root hyperplane $\alpha = 0$ is specified by the formula

$$s_\alpha v = v - \langle \alpha, v \rangle \alpha^\vee \ .$$

The sublattice $X_*(\widetilde{A})$ has the $\alpha^\vee$ in $\Delta^\vee$ as basis, and both lattices are stable under W. The partition of $X_*(A) \otimes \mathbb{R}$ by the affine root hyperplanes $\alpha = k$ is also $\mathcal{W}$-stable. If $\widetilde{\alpha}$ is the dominant root, the region

$$\mathfrak{A} = \{ v \in X_*(A) \otimes \mathbb{R} \mid \langle \alpha, v \rangle > 0 \text{ for all } \alpha \in \Delta, \langle \widetilde{\alpha}, v \rangle < 1 \}$$

is the interior of a fundamental domain for the group $\widetilde{W}$ generated by $X_*(\widetilde{A})$ and W. (Recall that T and $X_*(A)$ may be identified.) The walls of this **alcove** are labeled by

$$\widetilde{\Delta} = \Delta \cup \{ -\widetilde{\alpha} + 1 \} \ ,$$

the **affine roots** vanishing on its boundary. Let

$$\mathfrak{A}_\lambda = \overline{\mathfrak{A}} \cap \{ \lambda = 0 \} \ ,$$

the wall labelled by λ. If $\widetilde{S}$ is the set of affine reflections s_λ for λ in $\widetilde{\Delta}$, then $(\widetilde{W}, \widetilde{S})$ is a Coxeter system. In particular $\widetilde{S}$ generates $\widetilde{W}$.

The alcoves in $X_*(A) \otimes \mathbb{R}$ are the connected components of the partition by affine root hyperplanes. The group $\widetilde{W}$ acts transitively on them. The following is standard in the theory of Coxeter groups:

[dual] **3.1. Lemma.** *Suppose that w lies in $\widetilde{W}$. If $w\mathfrak{A}_\lambda \cap \overline{\mathfrak{A}} \neq \emptyset$ then $w = s_\lambda$.*

Every wall of an alcove is therefore $\widetilde{W}$-equivalent to a unique wall of $\mathfrak{A}$. This fails for the larger group $\mathcal{W}$. Any w be in $\mathcal{W}$ transforms $\mathfrak{A}$ into some other alcove, and there exists $\widetilde{w}$ in $\widetilde{W}$ transforming this in turn back to $\mathfrak{A}$. But the composite $\widetilde{w}^{-1}w$ will in general permute the walls, and in a well determined way.

[subgroup] **3.2. Proposition.** *The subgroup of T which takes $\mathfrak{A}$ into itself projects isomorphically onto Ω, so the exact sequences*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{T} & \longrightarrow & T & \longrightarrow & \Omega & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \widetilde{W} & \longrightarrow & \mathcal{W} & \longrightarrow & \Omega & \longrightarrow & 1
\end{array}
$$

split.

I'll identify Ω with this subgroup.

It would be a good idea, I think, to look at one example in detail. Suppose G to be $\mathrm{PGL}_3(k)$, so that $\widetilde{G} = \mathrm{SL}_3(k)$. The Weyl group is $\mathfrak{S}_3$. The group T may be identified with that of all diagonal matrices

$$
t = \varpi^m = \begin{bmatrix} \varpi^{m_1} & & \\ & \varpi^{m_2} & \\ & & \varpi^{m_3} \end{bmatrix}, \quad \left(m = (m_1, m_2, m_3) \right).
$$

modulo the scalar matrices with $m_1 = m_2 = m_3$. The subgroup $\widetilde{T}$ is that of ϖ^m with $m_1 + m_2 + m_3 = 0$, which embeds into PGL_3. The roots are, in multiplicative notation, the characters $\varpi^{m_i}/\varpi^{m_j}$ with $i \neq j$. The positive roots are, in additive notation:

$$
\begin{aligned}
\alpha_1 &= m_1 - m_2 \\
\alpha_2 &= m_2 - m_3 \\
\widetilde{\alpha} &= \alpha_1 + \alpha_2 = m_1 - m_3
\end{aligned}
$$

The group T has as basis the matrices

$$
\varepsilon_1 = \begin{bmatrix} \varpi & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \varepsilon_2 = \begin{bmatrix} \varpi & & \\ & \varpi & \\ & & 1 \end{bmatrix}
$$

and the coroot vectors are

$$
\alpha_1^\vee = \begin{bmatrix} \varpi & & \\ & \varpi^{-1} & \\ & & 1 \end{bmatrix}, \quad
\alpha_2^\vee = \begin{bmatrix} 1 & & \\ & \varpi & \\ & & \varpi^{-1} \end{bmatrix}, \quad
\widetilde{\alpha}^\vee = \begin{bmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{bmatrix}.
$$

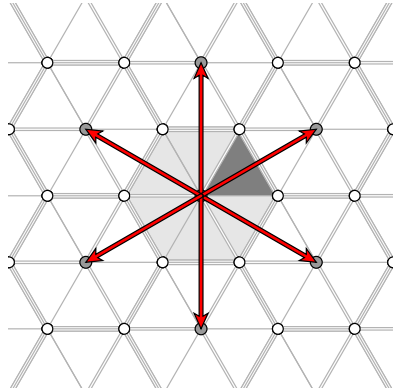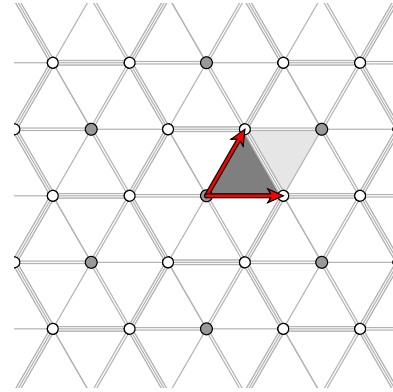These are shown in the following figures:

Figure 1. Coroots

Figure 2. Weights

The generators of $\widetilde{\mathcal{W}}$ in $\widetilde{S}$ are

$$
s_1 = \begin{bmatrix} & -1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad
s_2 = \begin{bmatrix} 1 & & \\ & & -1 \\ & 1 & \end{bmatrix}, \quad
s_3 = \begin{bmatrix} & & -\varpi^{-1} \\ & 1 & \\ \varpi & & \end{bmatrix}.
$$

In the figures on the left are the alcoves $s_i C$:
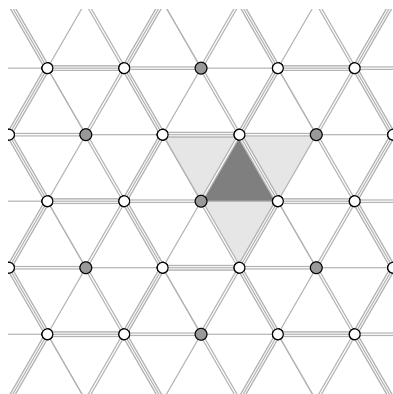
Figure 3. The $s_i(\mathfrak{A})$

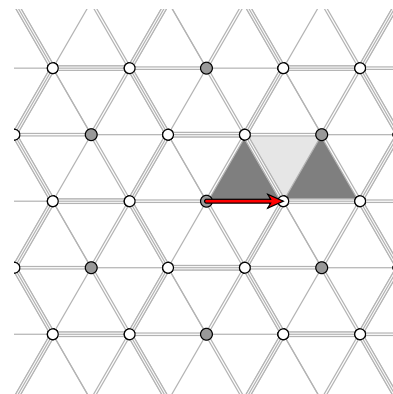**Figure 4.** ε_1 changes labels.

The index of $\widetilde{T}$ in T is 3, so Ω is the cyclic group of order 3, which in its action on the walls of $\mathfrak{A}$ just permutes them cyclically. The figure above on the right shows immediately that $\varepsilon_1(\mathfrak{A}) = s_3 s_1(\mathfrak{A})$ and Ω is generated by $s_1 s_3 \varepsilon_1$.

To return to the general discussion—any ω in Ω normalizes the Iwahori subgroup I. The group G is the disjoint union of the cosets IwI with w in $\mathcal{W}$, and if $w = \omega\tilde{w}$ then $IwI = I\omega I \cdot I\tilde{w}I$. The subalgebra of $\mathcal{H}(G, I)$ generated by the IsI with s in $\widetilde{S}$ is isomorphic to $\mathcal{H}(\widetilde{G}, \widetilde{I})$, that described by Macdonald. For s in $\widetilde{S}$, $\tau_s = \mathfrak{char}_{IsI}$ satisfies a relation

$$(\tau_s - q_s)(\tau_s + 1) = 0$$

for some positive integer constant q_s and is hence invertible. The length $\ell(w)$ of w in $\mathcal{W}$ is the length of its minimal expression as a product of elements of S; if $\ell(xy) = \ell(x) + \ell(y)$ then $IxyI = IxI \cdot IyI$. The operators $I\omega I$ with ω in Ω are trivially invertible since ω normalizes I. Hence:

[iwahori] **3.3. Lemma.** *If I is an Iwahori subgroup of G then each of its basis elements $\tau_x = \mathfrak{char}_{IxI}$ is invertible.*

We know that I possesses a factorization $I = N(\mathfrak{o})M(\mathfrak{o})\overline{N}(\mathfrak{p})$. For m in M^{--}

$$mN(\mathfrak{o})m^{-1} \subseteq N(\mathfrak{o}), \quad \overline{N}(\mathfrak{p}) \subseteq m\overline{N}(\mathfrak{p})m^{-1} .$$

In these circumstances

$$\delta_P(m) = \frac{1}{\left| N(\mathfrak{o})/mN(\mathfrak{o})m^{-1} \right|} .$$

[anym] **3.4. Proposition.** *For m in M^{--}, v in V^I*

$$\pi(\tau_m)v = \pi(m) \sum_{m^{-1}N(\mathfrak{o})m/N(\mathfrak{o})} \pi(n)v .$$

Proof. Essentially by definition

$$\pi(\tau_m)v = \sum_{y \in ImI/I} \pi(y)v .$$

The maps taking x to xmI induces a bijection of $I/I \cap mIm^{-1}$ with ImI/I. So we can write

$$\pi(m)v = \sum_{x \in I/I \cap mIm^{-1}} \pi(xm)v .$$

Because of the Iwahori factorization, inclusion induces a bijection of the quotient $N(\mathfrak{o})/mN(\mathfrak{o})m^{-1}$ with $I/I \cap mIm^{-1}$. We can again rewrite

$$\pi(\tau_m)v = \sum_{N(\mathfrak{o})/mN(\mathfrak{o})m^{-1}} \pi(n)\pi(m)v ,$$

which leads to the Proposition in one step.

From this follows immediately:

[jacquet] 3.5. Lemma. *For m_1, m_2 in M^{--}*

$$\tau_{m_1}\tau_{m_2} = \tau_{m_1 m_2} \; .$$

From these Lemmas, at least a rough version of Macdonald's formula follows.

[function] 3.6. Proposition. *The function Φ_χ restricted to M^{--} is M-finite.*

That is to say that it is contained in a finite-dimensional M-stable space of functions on M.

Proof. Suppose m to be in M^{--}. Let $v = \varphi_\chi$, $\widetilde{v} = \varphi_{\chi^{-1}}$. Then

$$\langle \pi(\tau_m)v, \widetilde{v} \rangle = \sum_{y \in ImI/I} \pi(y)v$$

$$= \sum_{x \in I/I \cap mIm^{-1}} \pi(x)\pi(m)v$$

$$\langle \pi(m)v, \widetilde{v} \rangle = \frac{1}{|ImI/I|} \langle \pi(\tau_m)v, \widetilde{v} \rangle \; .$$

Let $V = \mathrm{Ind}(\chi)$. Because τ_m is an invertible operator and M^{--} generates M, the representation of the operators τ_m extends to one of M on V^I. At any rate, this proves the Proposition.

The next step is to determine precisely which M-finite functions occur in the formula for Φ_χ. We shall do this by interpreting the representation of M on V^I more satisfactorily.

4. Jacquet modules

Suppose (π, V) to be any smooth representation of the unipotent subgroup N with unipotent radical N. I define the subspace $V(N)$ to be that spanned by all vectors u of the form $\pi(n)v - v$ for v in V, n in N. The quotient $V_N = V/V(N)$ is the maximal quotient on which N acts trivially, and an N-covariant linear map from V to any other N-trivial module will factor through it. Now assume that V is a smooth module over the parabolic subgroup P. The reductive group $M = P/N$ acts on the space V_N, defining the **Jacquet module** of π associated to P. As with parabolic induction, it is convenient to normalize this construction, so the Jacquet module is the canonical representation of M on $V/V(N)$ twisted by $\delta_P^{-1/2}$. Explicitly, if u in V has image v in V_N then $\pi_N(m)v$ is the image of $\delta_P^{-1/2}(m)\pi(m)u$.

[ggg] **4.1. Lemma.** *A vector in V lies in $V(N)$ if and only if*

$$\int_U \pi(u)v \, du = 0$$

for one or, equivalently, any large compact open subgroup U of N.

Proof. This is easy enough to see, since N has arbitrarily large compact open subgroups U. If $v = \pi(n)u - u$ we can find U containing n, and for any such U the integral vanishes. Conversely, if the integral vanishes we can express v as a sum of $\pi(n)u - u$ with n in U. ◻

[aaa] **4.2. Proposition.** *If*
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of smooth representations of P then

$$0 \longrightarrow A_N \longrightarrow B_N \longrightarrow C_N \longrightarrow 0$$

is exact as well.

This follows immediately from the Lemma.

The map Ω_1 taking f in the space $\mathrm{Ind}(\chi)$ of teh induced representation to $f(1)$ satisfies
$$\langle \Omega_1, R_p f \rangle = \delta_P^{1/2} \chi(p) \langle \Omega_1, f \rangle \ .$$

If $F \colon V \longrightarrow \mathrm{Ind}(\chi)$ is G-covariant, then

$$\langle \Omega_1, F(\pi(p)v) \rangle = \delta_P^{1/2} \chi(p) \langle \Omega_1, F(v) \rangle \ ,$$

so that composition with Ω_1 hence induces an M-covariant map from V_N to $\mathbb{D}_\chi$. This leads easily to:

[bbb] 4.3. Proposition. (Frobenius reciprocity) *If (π, V) is an arbitrary admissible representation of G then composition with Ω_1 induces a canonical isomorphism*

$$\operatorname{Hom}_G(V, \operatorname{Ind}(\chi)) \cong \operatorname{Hom}_M(V_N, \chi) \ .$$

Explicitly, if Ω is a P-covariant map from V_N to $\mathbb{D}_\chi$ then $v \mapsto \langle \Omega, R_g v \rangle$ maps V to $\operatorname{Ind}(\chi)$.

It was principally to simplify the formulation of this Proposition that the normalizing factor $\delta_P^{1/2}(p)$ was incorporated into the definition of π_N.

For any admissible representation (π, V), there is an intimate relationship between the space of vectors in V fixed by the Iwahori subgroup I and the subspace of vectors fixed by $M(\mathfrak{o})$ in the Jacquet module V_N. Recall that $\tau_m = \mathfrak{char}_{ImI}$ for m in M^{--}.

[ccc] 4.4. Proposition. *If v in V^I has image u in V_N, then for m in M^{--} the image of $\pi(\tau_m)v$ in V_N is equal to*

$$\delta_P^{-1/2}(m)\pi_N(m)u \ .$$

Proof. From an earlier calculation

$$\pi(\tau_m)v = \pi(m) \sum_{m^{-1}N(\mathfrak{o})m/N(\mathfrak{o})} \pi(n)v$$

which has image $\delta_P^{-1/2}(m)\pi_N(m)u$ in V_N. □

[ddd] 4.5. Proposition. *If V is any admissible representation of G then the canonical projection from V^I to $V_N^{M(\mathfrak{o})}$ is an isomorphism.*

Proof. First to be shown that it is an injection. From the previous calculation

$$\pi(\tau_m)v = \pi(m) \sum_{m^{-1}N(\mathfrak{o})m/N(\mathfrak{o})} \pi(n)v$$

for m in M^{--}. On the other hand, v lies in $V(N)$ if and only if

$$\sum_{U/U \cap I} \pi(u)v = 0$$

for some large subgroup U of N. Choose m in M^{--} so that $U \subseteq m^{-1}N(\mathfrak{o})m$. Since τ_m is invertible, v lies in $V(N)$ only if $v = 0$.

Next that it is a surjection. Suppose u in $V_N^{M(\mathfrak{o})}$, and choose v in $V^{M(\mathfrak{o})}$ whose image in V_N is u. Suppose that v is fixed by a (possibly very small) compact open subgroup

$\overline{N}_*$ of $\overline{N}$. Choose a in A^{--} such that $\overline{N}(\mathfrak{p}) \subseteq a\overline{N}_* a^{-1}$. Then $v_* = \delta_P^{-1}(a)\pi(a)v$ is fixed by $M(\mathfrak{o})\overline{N}(\mathfrak{p})$ and has image $\delta_P^{-1/2}(a)\pi_N(a)u$ in V_N. The average of $\pi(n)v_*$ over n in $N(\mathfrak{o})$ is the same as the average of the $\pi(x)v_*$ over x in I. This average v_{**} lies in V^I and has image $\delta_P^{-1/2}(a)\pi_N(a)u$ in V_N. But τ_a is invertible on V^I. The image of $\pi(\tau_a)^{-1}v_{**}$ in V_N is u.

One immediate consequence:

[fff] 4.6. Corollary. *Any irreducible unramified representation of G embeds into some* $\mathrm{Ind}(\chi)$*, and in particular its subspace of vectors fixed by K has dimension one.*

This is because of Frobenius reciprocity for representations induced from P.

These clever arguments originated with Jacquet, and were first presented in his Montecatini lectures [Jacquet:1971]. They can be extended to prove that the Jacquet module V_N is an admissible representation of M and furthermore that the Jacquet module controls the asymptotic behaviour of matrix coefficients of admissible representations. The best way to formulate this is:

[jacquetmoduke] 4.7. Theorem. *Suppose (π, V) to be any admissible representation of G. Let P be a parabolic subgroup of G with unipotent radical N, $\overline{P}$ a parabolic subgroup opposite to P. There exists a canonical isomorphism of the admissible dual of V_N with $\widetilde{V}_{\overline{N}}$ characterized by the property that for every v in V with image u in V_N and, $\widetilde{v}$ in $\widetilde{V}$ with image $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$ there exists $\varepsilon > 0$*

$$\langle \pi(a)v, \widetilde{v} \rangle = \delta_P^{1/2}(a)\langle \pi_N(a)u, \widetilde{u} \rangle_{\mathrm{can}}$$

whenever a in A^{--} satisfies the condition that $|\alpha(a)| < \varepsilon$ for all α in Σ_P^+.

A proof can be found in [Casselman:1974].

This result says that any matrix coefficient is asymptotically equal to an A-finite expression. The special thing about the spherical function is that this 'asymptotic' expression is valid for all a in A^{--}, as it is for all vectors fixed by the Iwahori subgroup I. What really distinguishes these, as we have seen, is that the operators τ_a are invertible on V^I. In general, if K is an open compact subgroup with an Iwahori factorization, then $\pi(\tau_a)$ for a in A^{--} will be invertible on a well-placed subspace of V^K, the image of the $\pi(\tau_a^n)$ for large n. It is this subspace which embeds into the Jacquet module.

We now know that the M-finite expression for Φ_χ is related to the structure of the Jacquet module of $\mathrm{Ind}(\chi)$. In the next section I'll describe that Jacquet module, but in the rest of this one I'll put in a short digression about unramified admissible representations.

It follows from the Corollary above that if (π, V) is an irreducible unramified representation then V^K has dimension 1. Recall that the characteristic homomorphism c_π from $\mathcal{H}(G/\!\!/K)$ to $\mathbb{D}$ is then well defined. It turns out that this distinguishes π.

[eee] **4.8. Proposition.** *Given any ring homomorphism c from $\mathcal{H}(G/\!/K)$ to $\mathbb{D}$, there exists up to isomorphism at most one irreducible, unramified, admissible representation whose characteristic homomorphism is c.*

Proof. The point is that the representation can be constructed directly from its characteristic homomorphism. Suppose (π, V) to be irreducible and unramified, c its characteristic homomorphism. Both V^K and $\widetilde{V}^K$ are one-dimensional. Fix $\widetilde{v} \neq 0$ in $\widetilde{V}$.

The map taking v to the function

$$\Phi_v = \langle \pi(g)v, \widetilde{v} \rangle$$

embeds V into the space $C^\infty(K\backslash G)$. For any f in $\mathcal{H}(G/\!/K)$ the operator

$$L_f = \int_G f(g) L_g \, dg$$

acts on $C^\infty(K\backslash G)$, commuting with the right regular representation R of G. Explicitly, for v in V

$$L_f \Phi_v = \int_G f(g) L_g \Phi_v \, dg$$

and

$$
\begin{aligned}
L_f \Phi_v(x) &= \int_G f(g) L_g \Phi_v(x) \, dg \\
&= \int_G f(g) \langle \pi(g^{-1}x)v, \widetilde{v} \rangle \, dg \\
&= \int_G f(g) \langle \pi(x)v, \widetilde{\pi}(g)\widetilde{v} \rangle \, dg \\
&= \langle \pi(x)v, \widetilde{\pi}(f)\widetilde{v} \rangle \\
&= c_{\widetilde{\pi}}(f) \langle \pi(x)v, \widetilde{v} \rangle \\
&= c_{\widetilde{\pi}}(f) \Phi_v(x) \ .
\end{aligned}
$$

Note that since

$$\pi(f)v = c_\pi(f)v = \int_G f(g)\pi(g)v \, dg$$

for all f in the Hecke algebra,

$$\langle v, \widetilde{\pi}(f)\widetilde{v} \rangle = \int_G f(g)\langle v, \widetilde{\pi}(g)\widetilde{v} \rangle \, dg = \int_G f(g^{-1})\langle \widetilde{\pi}(g)v, \widetilde{v} \rangle \, dg$$

and $c_{\widetilde{\pi}}(f) = c_\pi(f^\vee)$ where $f^\vee(g) = f(g^{-1})$.

Define $\mathcal{L}_{\widetilde{c}}$ to be the space of all smooth functions Φ on $K\backslash G$ such that

$$L_f \Phi = c_{\widetilde{\pi}}(f)\Phi$$

for all f in the Hecke algebra.

I claim now that the image of V is the unique irreducible G-stable subspace of $\mathcal{L}_{\tilde{c}}$.

To see this: • every G-stable subspace of $C^{\infty}(K\backslash G)$ contains a function fixed by K; • the space of functions in $\mathcal{L}_{\tilde{c}}$ fixed by K has dimension 1. I leave these both as exercises. ◻

5. The Jacquet module of $\mathrm{Ind}(\chi)$

The group G possesses a decomposition into disjoint double cosets PwP indexed by elements of the Weyl group W. There is a single one of these which is an open subset of G, the one with $w = w_\ell$, the longest element of W. There is exactly one which is closed in G, that with $w = 1$, where $PwP = P$ itself. The closure of a coset PwP is the union of cosets PxP where $x \leq w$ in what is frequently called the **Bruhat order** on the Weyl group W. The union of the subsets PxP with $w \leq x$ is then an open neighbourhood of PwP in G.

The Bruhat order has a combinatorial description—if w has the **reduced decomposition** $w = s_1 \ldots s_n$ as a minimal product of elementary reflections, then $x \leq w$ if and only if x can be expressed as a product of a subsequence of the s_i in the same order.

The Bruhat order on W induces a P-stable filtration on $\mathcal{I} = \mathrm{Ind}(\chi)$. For each w in the Weyl group W, define the space $\mathcal{I}_w$ to be that of f in $\mathcal{I}$ with support on the union of the PxP with $w^{-1} \leq x$. Thus $\mathcal{I} = \mathcal{I}_1$, for example, and $\mathcal{I}_{w_\ell}$ is contained in all others. Each space $\mathcal{I}_w$ is stable under P. If f lies in $\mathcal{I}_w$ then its restriction to $Pw^{-1}P$ is smooth and of compact support modulo the left factor P. It satisfies the equation $f(px) = \delta_P^{1/2}(p)\chi(p)f(x)$ for all x in $Pw^{-1}P$. The space of all such restrictions is as a representation of N isomorphic to $C_c^\infty(N \cap wNw^{-1}\backslash N)$. For f in $\mathcal{I}_w$, the integral

$$\langle \Omega_w, f \rangle = \int_{N \cap wNw^{-1}\backslash N} f(w^{-1}n)\, dn$$

is hence a finite sum, defining an N-invariant linear functional on $\mathcal{I}_w$. An easy calculation shows that in addition

$$\langle \Omega_w, R_m f \rangle = \delta_P^{1/2}(m)\chi(w^{-1}mw)\,\langle \Omega_w, f \rangle = \delta_P^{1/2}(m)\,w\chi(m)\,\langle \Omega_w, f \rangle$$

for all m in M, so that Ω_w induces an M-covariant map from the Jacquet module of $\mathcal{I}_w$ with respect to N to $\mathbb{D}_{w\chi}$. Let

$$J_w = I_w \Big/ \sum_{w^{-1} < x} I_x\ ,$$

also a representation of P. If f lies in one of the $\mathcal{I}_x$ with $x > w^{-1}$ then $\langle \Omega_w, f \rangle = 0$, so Ω_w factors through J_w.

[hhh] **5.1. Lemma.** *The linear functional Ω_w induces an isomorphism of $\mathbb{D}_{w\chi}$ and the Jacquet module of J_w with respect to N.*

This is because the integral is, up to scalar multiples, the only N-invariant functional on $C_c^\infty(N, \mathbb{D})$.

Because the functor $V \rightsquigarrow V_N$ is exact:

[filtration] 5.2. Theorem. *The Bruhat filtration of $\mathrm{Ind}(\chi)$ induces a filtration of its Jacquet module. The graded module associated to this filtration is the direct sum of the $\mathbb{D}_{w\chi}$.*

If χ is regular, which is to say $w\chi \neq \chi$ for all w in W, then all the extensions occurring in this filtration must split, and the Jacquet module of $\mathrm{Ind}(\chi)$ is itself the direct sum of the $\mathbb{D}_{w\chi}$. In other words, in these circumstances Ω_w extends to define a P-covariant map from all of $\mathrm{Ind}(\chi)$ to $\mathbb{D}_{\delta_P^{1/2} w\chi}$.

To summarize, we now know that

$$\Phi_\chi(m) = \delta_P^{1/2}(m) \sum_{w \in W} \alpha_w(\chi)\, w\chi(m)$$

for m in M^{--} and regular χ. It is also easy to see that dealing with non-regular χ is a matter of applying an algebraic version of l'Hôpital's rule. The task remaining is to calculate the coefficients α_w.

6. The operators T_w

Finding a formula for the the $\alpha_w(\chi)$ comes down to doing this for just one of them. More precisely, we'll see that $\alpha_w(\chi) = \alpha_1(w\chi)$.

According to Frobenius reciprocity for $\mathrm{Ind}(\chi)$, the linear functional Ω_w corresponds to a G-covariant map

$$T_w\colon \mathrm{Ind}(\chi) \longrightarrow \mathrm{Ind}(w\chi), \quad f \longmapsto \langle \Omega_w, R_g f \rangle \, .$$

[tw] **6.1. Proposition.** *If* $\ell(xy) = \ell(x) + \ell(y)$ *then*

$$T_x T_y = T_{xy}$$

Proof. The length $\ell(w)$ of w is the length of a reduced expression for it, in terms of elementary reflections. It is also the cardinality of the root set

$$\Lambda_w = \{\alpha > 0 \mid w^{-1}\alpha < 0\} \, .$$

If x and y are two elements of W with $\ell(xy) = \ell(x) + \ell(y)$ then Λ_{xy} is the disjoint union of $x\Lambda_y$ and Λ_x.

The decomposition of

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{n}_\alpha = \sum_{\alpha>0,\, w^{-1}\alpha>0} \mathfrak{n}_\alpha \quad + \sum_{\alpha>0,\, w^{-1}\alpha<0} \mathfrak{n}_\alpha$$

corresponds to the factorization

$$N = (wNw^{-1} \cap N)(w\overline{N}w^{-1} \cap N) = N_w^+ N_w^-$$

so that

$$N_w^+ \backslash N \cong N_w^- \, .$$

When $\ell(xy) = \ell(x) + \ell(y)$ the decomposition $\Lambda_{xy} = x\Lambda_y \cup \Lambda_y$ corresponds to a direct sum decomposition

$$\mathfrak{n}_{xy}^- = \mathrm{Ad}(x)\, \mathfrak{n}_y^- + \mathfrak{n}_x^-$$

and a factorization

$$N_{xy}^- = xN_y^- x^{-1} \cdot N_x^- \, .$$

Thus

$$
\begin{aligned}
T_x T_y f(g) &= \int_{N_x^-} T_y f(x^{-1} n_x g)\, dn_x \\
&= \int_{N_x^-} \int_{N_y^-} f(y^{-1} n_y x^{-1} n_x g)\, dn_y\, dn_x \\
&= \int_{N_x^-} \int_{N_y^-} f(y^{-1} x^{-1} x n_y x^{-1} n_x g)\, dn_y\, dn_x \\
&= \int_{N_{xy}^-} f(y^{-1} x^{-1} n_{xy} g)\, dn_{xy}
\end{aligned}
$$

since $(n_y, n_x) \longrightarrow x n_y x^{-1} n_x$ is a measure-preserving bijection between $N_y^- \times N_x^-$ and N_{xy}^-.

Since the subspace of $\mathrm{Ind}(\chi)$ of vectors fixed by K has dimension one,

$$
T_w \varphi_\chi = \zeta_w(\chi) \varphi_{w\chi}
$$

for some scalar $\zeta_w(\chi)$. The Proposition leads to a product formula for $\zeta_w(\chi)$ that it is not yet convenient to display. But at any rate it should be clear that in principle the problem of evaluating it for an arbitrary w reduces to evaluating it in the special case when w is an elementary reflection s. This will be done in the next section, in a self-contained calculation. One consequence of that computation is that T_w is generically an isomorphism. Because of this:

- As a function of χ, the spherical function is W-invariant: $\Phi_{w\chi} = \Phi_\chi$;
- The Satake transform has its image in $\mathcal{H}(M // M(\mathfrak{o}))^W$;
- For regular χ, $\alpha_w(\chi) = \alpha_1(w\chi)$.

The calculation in the next section will also show that $\zeta_s(\chi)$ is of the form $C\big(\chi(a_\alpha)\big)$ where $C(X)$ is an explicitly calculated rational function of X. Here s is the elementary reflection corresponding to the root α in Δ_P. The Proposition in this section then leads to the product formula

$$
T_w \varphi_\chi = \prod_{\alpha > 0,\, w^{-1}\alpha < 0} C\big(\chi(a_\alpha)\big)\, \varphi_{w\chi}.
$$

This allows us also to evaluate the functionals $\langle \Omega_w, \varphi_\chi \rangle$. And this in turn, as I shall next explain, will give us Macdonald's formula at last.

7. Asymptotics

I recall that for all m in M^{--} and regular χ

$$\Phi_\chi(m) = \delta_P^{1/2}(m) \sum_w \alpha_1(w\chi)\, w\chi(m) = \delta_P^{1/2}(m) \sum_w \alpha_{w_\ell}(w_\ell^{-1} w\chi)\, w\chi(m) \ .$$

Macdonald's formula follows directly from this:

[asymptotics] 7.1. Proposition. *We have*

$$\alpha_{w_\ell}(\chi) = \mu_G \left\langle \Omega_{w_\ell}, \varphi_\chi \right\rangle = \mu_G \prod_{\alpha > 0} C\big(\chi(a_\alpha)\big) \ .$$

The proof of this will require still one more digression. We have defined the linear functionals Ω_w on $\mathrm{Ind}(\chi)$, at least formally, by the integral

$$\left\langle \Omega_w, f \right\rangle = \int_{N \cap wNw^{-1} \backslash N} f(w^{-1} n)\, dn \ .$$

The rigourous interpretation of this is that the integral is well defined on $\mathcal{I}_w$ and extends uniquely to a P-covariant map onto $\mathbb{D}_{\delta_P^{1/2} w\chi}$.

The Bruhat decomposition $G = \bigcup PwP$ gives rise to the decomposition

$$G = \bigcup_w PwPw_\ell = \bigcup_w Pww_\ell\, w_\ell^{-1} Pw_\ell = \bigcup_w Pww_\ell \overline{P} \ .$$

The filtration on $\mathrm{Ind}(\chi^{-1})$ produced by this will also be indexed by W, but according to the order opposite to the Bruhat order. Thus $P\overline{P} = Pw_\ell Pw_\ell$ is the largest (and open) coset, $Pw_\ell \overline{P} = Pw_\ell$ the smallest. For regular χ the linear functionals

$$\left\langle \widetilde{\Omega}_w, \widetilde{f} \right\rangle = \int_{\overline{N} \cap wNw^{-1} \backslash \overline{N}} f(w^{-1} \overline{n})\, d\overline{n}$$

identify the Jacquet module of $\mathrm{Ind}(\chi^{-1})$ with the direct sum $\bigoplus \mathbb{D}_{w\chi^{-1}}$. In particular

$$\left\langle \widetilde{\Omega}_{w_\ell}, \widetilde{f} \right\rangle = f(w_\ell) \ .$$

From the general result about asymptotic behaviour of matrix coefficients we deduce that for suitable constants β_w

$$\left\langle R_a f, \widetilde{f} \right\rangle = \delta_P^{1/2}(a) \sum_w \beta_w\, w\chi(a) \left\langle \Omega_w, f \right\rangle \left\langle \widetilde{\Omega}_w, \widetilde{f} \right\rangle$$

for all f in $\mathrm{Ind}(\chi)$, $\widetilde{f}$ in $\mathrm{Ind}(\chi^{-1})$ and a 'small enough'. The real point here is that the coefficients β_w are independent of f and $\widetilde{f}$. Therefore in order to know what α_{w_ℓ} is, we have now to consider the asymptotic behaviour just for a particular f and $\widetilde{f}$.

Choose f and $\widetilde{f}$ with support in the open set $Pw_\ell N$. Then

$$\langle R_a f, \widetilde{f} \rangle = \mu_G \int_N f(w_\ell na)\widetilde{f}(w_\ell n)\, dn$$

as we have seen in defining μ_G. We can rewrite the integral as

$$\int_N f(w_\ell a w_\ell^{-1}\, w_\ell\, a^{-1}na)\widetilde{f}(w_\ell n)\, dn$$

$$= \delta_P^{-1/2}(a)\, w_\ell\chi(a) \int_N f(w_\ell\, a^{-1}na)\widetilde{f}(w_\ell n)\, dn$$

$$= \delta_P^{1/2}(a)\, w_\ell\chi(a) \int_N f(w_\ell n)\widetilde{f}(w_\ell ana^{-1}\, dn$$

$$= \delta_P^{1/2}(a)\, w_\ell\chi(a)\, \widetilde{f}(w_\ell) \int_N f(w_\ell n)\, dn$$

if a is 'small enough'. Hence $\beta_{w_\ell} = \mu_G$. From this it follows that $\beta_w = \mu_G$ for all w.

8. Rank one groups

Corresponding to each simple positive root α is a parabolic subgroup $P_{\{\alpha\}}$ whose Levi factor $M_{\{\alpha\}}$ has semi-simple rank one. The group $M_{\{\alpha\}}$ contains M, and the image of P modulo the unipotent radical of $P_{\{\alpha\}}$ is a minimal parabolic subgroup of $M_{\{\alpha\}}$. The representation $\mathrm{Ind}(\chi)$ of G is equal to a representation induced in two steps, from P to $P_{\{\alpha\}}$ and thence from $P_{\{\alpha\}}$ to G. The intertwining operator T_{s_α} is likewise induced from an operator between representations of $M_{\{\alpha\}}$. The factor $c_{s_\alpha}(\chi)$ for G is the same as that for $M_{\{\alpha\}}$.

So now, for the moment, we are reduced to the following question: Suppose G to be a reductive group over k of semi-simple rank one, w the non-trivial element in the Weyl group. Suppose χ to be a regular unramified character of M. We know that $T_w \varphi_\chi = \zeta_w(\chi)\varphi_{w\chi}$. What is $\zeta_w(\chi)$? Equivalently, what is $\langle \Omega_w, \varphi_\chi \rangle$?

The calculation reduces immediately to one on the simply connected cover of G, so we may as well assume G to be simply connected, hence of the form $\mathrm{SL}_2(E)$ or $\mathrm{SU}_3(E)$. I may as well assume $E = k$, too.

In both cases, we want to evaluate $\langle \Omega_w, \varphi_\chi \rangle$, where φ_χ is defined by the formula

$$\varphi_\chi(nmk) = \delta_P^{1/2}(m)\chi(m) .$$

In the published literature, the coefficient field $\mathbb{D}$ is always $\mathbb{C}$, and the usual calculation proceeds in two stages: (i) calculation of a convergent series in the region where the integral defining Ω_w converges and (ii) analytic continuation. With our self-imposed handicap, this is not an allowable option. It is not difficult to think up a way to deal with this problem, however. For any m in M, the function $\psi = \left(R_m - \delta_P^{1/2}(m)\chi(m)\right)\varphi_\chi$ is 0 at 1 and has support on the coset PwP. Hence $\langle \Omega_w, \psi \rangle$ can be evaluated as a finite sum, which we shall calculate separately in each case. On the other hand

$$\begin{aligned}
\langle \Omega_w, \psi \rangle &= \langle \Omega_w, \left(R_m - \delta_P^{1/2}(m)\chi(m)\right)\varphi_\chi \rangle \\
&= \langle \Omega_w, R_m\varphi_\chi \rangle - \delta_P^{1/2}(m)\chi(m)\langle \Omega_w, \varphi_\chi \rangle \\
&= \delta_P^{1/2}(m)\chi^{-1}(m)\langle \Omega_w, \varphi_\chi \rangle - \delta_P^{1/2}(m)\chi(m)\langle \Omega_w, \varphi_\chi \rangle \\
&= \delta_P^{1/2}(m)\left(\chi^{-1}(m) - \chi(m)\right)\langle \Omega_w, \varphi_\chi \rangle \\
\langle \Omega_w, \varphi_\chi \rangle &= \frac{\delta_P^{-1/2}(m)}{\chi^{-1}(m) - \chi(m)}\langle \Omega_w, \psi \rangle .
\end{aligned}$$

• Let $G = \mathrm{SL}_2(k)$, $K = \mathrm{SL}_2(\mathfrak{o})$, P be the group of upper triangular invertible matrices, $M = A$ be the group of diagonal invertible matrices. Further let

$$w = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad w^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix},$$

which represents the single non-trivial element in the Weyl group. Let

$$\chi: \begin{bmatrix} x & \\ & x^{-1} \end{bmatrix} \longmapsto |x|^s$$

be an unramified character of A, and let

$$\omega = \begin{bmatrix} \varpi & \\ & \varpi^{-1} \end{bmatrix} \quad \text{so that} \quad \delta_P^{1/2}(\omega) = q^{-1}, \quad \chi(\omega) = q^{-s} .$$

What we find here is that

$$
\begin{aligned}
\langle \Omega_w, \varphi_\chi \rangle &= \frac{\delta_P^{-1/2}(\omega)}{\chi^{-1}(\omega) - \chi(\omega)} \langle \Omega_w, \psi \rangle \\
&= \frac{q}{q^s - q^{-s}} \langle \Omega_w, \psi \rangle \\
&= \frac{q^{1-s}}{(1 - q^{-s})(1 + q^{-s})} \langle \Omega_w, \psi \rangle \\
&= \frac{q^{1-s}}{(1 - q^{-s})(1 + q^{-s})} \int_N \psi(w^{-1}n) \, dn \\
&= \frac{q^{1-s}}{(1 - q^{-s})(1 + q^{-s})} \int_k \psi \left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \right) dx
\end{aligned}
$$

where $\mathrm{meas}(\mathfrak{o}) = 1$. The first step in evaluating the integral is to find a formula for

$$\psi(w^{-1}n) = \psi \left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \right) .$$

By definition

$$
\begin{aligned}
\psi(w^{-1}n) &= \big(R_\omega - \delta_P^{1/2}(\omega)\chi(\omega) \big) \varphi_\chi(w^{-1}n) \\
&= \varphi(w^{-1}n\omega) - \delta_P^{1/2}(\omega)\chi(\omega)\varphi_\chi(w^{-1}n) .
\end{aligned}
$$

For an arbitrary n in N we have

$$\varphi_\chi(w^{-1}n\omega) = \varphi_\chi(w^{-1}\omega w \cdot w^{-1} \cdot \omega^{-1}nt) = \delta_P^{-1/2}(\omega)\chi^{-1}(\omega)\varphi_\chi(w^{-1} \cdot \omega^{-1}n\omega)$$

so that we must next find a formula for $\varphi_\chi(w^{-1}n)$ for an arbitrary $n = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$ in N.

If x lies in $\mathfrak{o}$ then wn lies in K and $\varphi_\chi(wn) = 1$. Otherwise, we must factor the matrix

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} = \begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix}$$

according to $G = NAK$. I could just write out the factorization directly, but it is probably a good idea to show where it comes from. First I note that it suffices to find such a factorization for

$$w^{-1}nw = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & \\ -x & 1 \end{bmatrix}$$

since w lies in K. Now the matrix

$$\begin{bmatrix} 1 & \\ -x & 1 \end{bmatrix}$$

lies in $\overline{N} - \{1\}$, and we know that it can be factored as $n_1 a w n_2$ according to 2.6.6 of Macdonald's book, at least in principle. But this factorization is practical in the sense that it can be found by applying a mild variant of the familiar Gauss elimination algorithm. We obtain

$$\begin{bmatrix} 1 & \\ -x & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} x^{-1} & \\ & x \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & -x^{-1} \\ & 1 \end{bmatrix}$$

But as long as x doesn't lie in $\mathfrak{o}$, its inverse x^{-1} will, so the last two factors lie in K. In fact, x^{-1} will lie in $\mathfrak{o}$ as long as x doesn't lie in $\mathfrak{p}$, so:

[eval] **8.1. Lemma.** *We have*

$$\varphi_\chi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right) = \begin{cases} 1 & \text{if } x \in \mathfrak{p} \\ |x|^{-1-s} & \text{otherwise} \end{cases}$$

Therefore

$$\psi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right)$$

$$= \delta_P^{-1/2}(\omega)\chi^{-1}(\omega)\varphi_\chi \left(\begin{bmatrix} & 1 \\ -1 & -\varpi^{-2}x \end{bmatrix} \right) - \delta_P^{1/2}(\omega)\chi(\omega)\varphi_\chi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right)$$

$$= q^{1+s}\varphi_\chi \left(\begin{bmatrix} & 1 \\ -1 & -\varpi^{-2}x \end{bmatrix} \right) - q^{-1-s}\varphi_\chi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right)$$

so that

$$\psi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right) = \begin{cases} q^{1+s} - q^{-1-s} & \text{if } x \in \mathfrak{p}^2 \\ 1 - q^{-1-s} & \text{if } x \in \mathfrak{p} - \mathfrak{p}^2 \\ 0 & \text{if } x \notin \mathfrak{p} \,. \end{cases}$$

Then

$$\int_k \psi \left(\begin{bmatrix} & 1 \\ -1 & -x \end{bmatrix} \right) dx = q^{-2}(q^{1+s} - q^{-1-s}) + (q^{-1} - q^{-2})(1 - q^{-1-s})$$

$$= q^{-1}(1 + q^s)(1 - q^{-1-s}) \,.$$

Finally

$$\langle \Omega_w, \varphi_\chi \rangle = \frac{q^{1-s}}{(1 - q^{-s})(1 + q^{-s})} \langle \Omega_w, \psi \rangle$$

$$= \frac{q^{1-s}}{(1 - q^{-s})(1 + q^{-s})} q^{-1}(1 + q^s)(1 - q^{-1-s})$$

$$= \frac{1 - q^{-1-s}}{1 - q^{-s}}$$

$$= \frac{1 - q^{-1}\chi(\omega)}{1 - \chi(\omega)} .$$

• Now let ℓ/k be an unramified quadratic extension, and let G be the unitary group of the Hermitian form $x_1\overline{x}_3 + x_2\overline{x}_2 + \overline{x}_1 x_3$ with matrix

$$J = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$$

In other words, G is the group of all X in $\mathrm{GL}_3(\ell)$ such that

$$^tX \, J \, \overline{X} = J \quad \text{or} \quad \overline{X} = J \, {}^tX^{-1} \, J .$$

Let

$$P = \text{upper triangular matrices in } G$$

$$M = \text{diagonal matrices in } G$$

$$w = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$$

$$\omega = \begin{bmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{bmatrix}$$

Thus a typical element in M looks like

$$\begin{bmatrix} z & & \\ & \overline{z}/z & \\ & & \overline{z}^{-1} \end{bmatrix}$$

and one in N looks like

$$\begin{bmatrix} 1 & x & y \\ & 1 & \overline{x} \\ & & 1 \end{bmatrix}, \quad y + \overline{y} = x\overline{x} .$$

The element w represents the non-trivial element in the Weyl group, and

$$\delta_P^{1/2}(\omega) = q^{-2} \ .$$

We want to calculate $\langle \Omega_\chi, \varphi_\chi \rangle$, and as before the crucial step is to evaluate $\varphi_\chi(w^{-1}n)$ where n lies in N. It again comes down to finding the $NMwN$ factorization of $w^{-1}nw$, which is easy to do:

$$n = \begin{bmatrix} 1 & x & y \\ & 1 & \overline{x} \\ & & 1 \end{bmatrix}$$

$$w^{-1}nw = \begin{bmatrix} 1 & & \\ -\overline{x} & 1 & \\ y & -x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x/\overline{y} & 1/y \\ & 1 & -\overline{x}/y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1/\overline{y} & & \\ & \overline{y}/y & \\ & & y \end{bmatrix} \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 1 & x/\overline{y} & 1/y \\ & 1 & -\overline{x}/y \\ & & 1 \end{bmatrix} \ .$$

If $y \in \mathfrak{p}$ then since $x\overline{x} = -y + \overline{y}$, x will also be in $\mathfrak{p}$ and n will lie in K. Otherwise, x/y will lie in $\mathfrak{o}$ and the matrix

$$\begin{bmatrix} 1 & x/\overline{y} & 1/y \\ & 1 & -\overline{x}/y \\ & & 1 \end{bmatrix}$$

will lie in K.

Let χ be the character of M taking ω to q^{-s}. From the calculations above, we have

$$\varphi_\chi(w^{-1}n) = \begin{cases} 1 & \text{if } y \in \mathfrak{p} \\[2mm] |y|^{-2-s} & \text{otherwise} \end{cases}$$

A calculation only slightly more complicated than the one for SL_2 shows that

$$\begin{aligned} \langle \Omega_w, \varphi_\chi \rangle &= \frac{\delta_P^{-1/2}(\omega)}{\chi^{-1}(t) - \chi(\omega)} \langle \Omega_w, \psi \rangle \\[2mm] &= \frac{q^2}{q^s - q^{-s}} \langle \Omega_w, \psi \rangle \\[2mm] &= \frac{q^{2-s}}{1 - q^{-2s}} \langle \Omega_w, \psi \rangle \\[2mm] &= \frac{(1 - q^{-2-s})(1 + q^{-1-s})}{1 - q^{-2s}} \\[2mm] &= \frac{\left(1 - q^{-2}\chi(\omega)\right)\left(1 + q^{-1}\chi(\omega)\right)}{1 - \chi(\omega)^2} \ . \end{aligned}$$

This concludes the proof of Macdonald's formula for the spherical function.

9. The character

This section follows [van Dijk:1972].

The full Hecke algebra $\mathcal{H}(G)$ associated to G is the space of all smooth, compactly supported functions on G (with values in $\mathbb{D}$). Multiplication is by convolution. This algebra acts on the space of any admissible representation of G, and indeed admissible representations of G are easily seen to be equivalent to admissible modules over $\mathcal{H}(G)$.

If (π, V) is an admissible representation of G its **character** is a linear functional on $\mathcal{H}(G)$, taking f to $\mathrm{trace}\,\pi(f)$. It can in fact be represented by integration of a function on an open subset of G, and in this section I'll derive its formula when $\pi = \mathrm{Ind}(\chi)$. This formula and some of the results leading to it will play a role in the discussion of the Satake isomorphism in the next section, and other parts of this section will play a role in the later discussion of endoscopy.

If f lies in $\mathcal{H}(G)$ then for any φ in $\mathrm{Ind}(\chi)$

$$
\begin{aligned}
R_f \varphi(x) &= \int_G f(g)\varphi(xg)\,dg \\
&= \int_G f(x^{-1}g)\varphi(g)\,dg \\
&= \int_K dk \int_M \delta_P^{-1}(m)\,dm \int_N \varphi(nmk)f(x^{-1}nmk)\,dn \\
&= \int_K \varphi(k)\,dk \int_M \chi(m)\delta_P^{-1/2}(m)\,dm \int_N f(x^{-1}nmk)\,dn \; .
\end{aligned}
$$

The space $\mathrm{Ind}(\chi)$ may be identified as a linear space and even as a K-space with $\mathrm{Ind}(\chi \mid K \cap P, K)$, and acting on this space R_f has the kernel

$$
K_f(k, \ell) = \int_M \chi(m)\delta_P^{-1/2}(m)\,dm \int_N f(\ell^{-1}nmk)\,dn \; .
$$

The trace of R_f on $\mathrm{Ind}(\chi)$ is therefore

$$
\int_K dk \int_M \chi(m)\delta_P^{-1/2}(m)\,dm \int_N f(k^{-1}nmk)\,dn \; .
$$

For any f in $\mathcal{H}(G)$ define

$$
\overline{f}(x) = \int_K f(k^{-1}xk)\,dk \; ,
$$

that is to say the projection of f onto the functions invariant with respect to conjugation by K. This lies again in $\mathcal{H}(G)$ since if f is right invariant under the subgroup

$K_\bullet$ of K and u lies in $\bigcap_{k \in K} k K_\bullet\, k^{-1}$ (a finite intersection, hence an open, compact subgroup) then for any k in K we have $uk = k \cdot k^{-1}uk$ with $k^{-1}uk \in K_\bullet$ and

$$\begin{aligned}
\overline{f}(xu) &= \int_K f(k^{-1}xuk)\, dk \\
&= \int_K f(k^{-1}xk \cdot k^{-1}uk)\, dk \\
&= \int_K f(k^{-1}xk)\, dk \\
&= \overline{f}(x) \ .
\end{aligned}$$

Also for any f in $\mathcal{H}(G)$ define

$$f_P(m) = \delta_P^{-1/2}(m) \int_N f(nm)\, dn \ ,$$

which will be a function in $\mathcal{H}(M)$. The calculation of the trace of R_f can now be formulated like this:

[trace] **9.1. Proposition.** *For any f in $\mathcal{H}(G)$, the trace of R_f acting on $\mathrm{Ind}(\chi \mid P, G)$ is*

$$\int_M \chi(m)\overline{f}_P(m)\, dm \ .$$

If f lies in $\mathcal{H}(G/\!/K)$ then $\overline{f} = f$, the trace of R_f is just $c_\chi(f)$, and the integral is a simple sum. We recover a formula to be found in Satake's book on spherical functions:

[satakeint] **9.2. Corollary.** *For any f in $\mathcal{H}(G/\!/K)$*

$$c_\chi(f) = \sum_{M/M(\mathfrak{o})} \chi(m) f_P(m) \ .$$

If T is any maximal torus in G (not necessarily maximally split) then the adjoint action of T on $\mathfrak{g}/\mathfrak{t}$ (over the algebraic closure $\overline{k}$ of k) is a direct sum of one-dimensional subspaces on which T acts by characters called its **roots**. For any t in T let

$$D(t) = \det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t) - I) \ .$$

This function can also be defined for any semi-simple element of G, independently of a torus containing it, according to the recipe

$$\det(\mathrm{Ad}_{\mathfrak{g}}(t) - I + \lambda) = D(t)\lambda^r + \text{higher powers of } \lambda$$

where r is the rank of G over $\overline{k}$. The element t is called **regular** if $D(t) \neq 0$, or equivalently if the connected component of the centralizer of t is T. Let G^{reg} be the regular elements of G.

The function $D(t)$ is important because of this elementary formula:

[conjugationmap] 9.3. Lemma. *The conjugation map from $G \times T$ to G taking taking (g, t) to gtg^{-1} has differential*

$$\mathfrak{g} \oplus \mathfrak{t}\colon\ (x, y) \longmapsto \mathrm{Ad}(g)\big(\mathrm{Ad}(t^{-1}) - I)x + y\big)\ .$$

As a consequence, the map $G/T \times T^{\mathrm{reg}} \to G$ taking (g, t) to gtg^{-1} has open image, the regular elements of G that are conjugate to an element of T, and we have this fundamental integral formula:

[orbitalint] 9.4. Lemma. *For any f in $\mathcal{H}(G)$ with support in the open set G^{reg}*

$$\int_G f(g)\,dg = \sum_T \frac{1}{W_T} \int_T |D(t)|\,dt \int_{G/T} f(xtx^{-1})\,dx$$

where W_T is the finite quotient of $N_G(T)$ by T.

For $T = M$, we have

$$|D(t)| = \left| \prod_\Sigma \det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t) - I) \right|$$

$$= \left| \prod_{\alpha > 0} \det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t) - I) \det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t^{-1}) - I) \right|$$

$$= \delta_P^{-1}(t) \left| \prod_{\alpha > 0} \det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t) - I) \right|^2$$

$$\left| \prod_{\alpha > 0} \det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t) - I) \right| = |D(t)|^{1/2} \delta_P^{1/2}(m)\ .$$

The following originates in Harish-Chandra's work on real groups.

[hcl] 9.5. Proposition. Harish-Chandra's Lemma) *For regular t in M the map $N \to N$ taking n to $n \cdot tn^{-1}t^{-1}$ is a bijection with modulus*

$$|\det(\mathrm{Ad}_{\mathfrak{n}}(t) - I)|\ .$$

Proof. The group N possesses a filtration by normal subgroups with graded groups equal to the N_α. The modulus on each of these is easily seen to be $\det(\mathrm{Ad}_{\mathfrak{n}_\alpha}(t) - I)$. $\square$

[fbar] **9.6. Proposition.** *For any t in M and f in $\mathcal{H}(G)$*

$$\overline{f}_P(t) = |D(t)|^{1/2} \int_{G/M} f(xtx^{-1})\, dx \ .$$

The curious feature of this formula is that although the **orbital integral**

$$\int_{G/M} f(xtx^{-1})\, dx$$

may be badly behaved as t approaches the singular elements of G, the product of the orbital integral by the **normalizing factor** $|D(t)|^{1/2}$ is nicely behaved, since $\overline{f}_P(t)$ is.

Proof. By Harish-Chandra's Lemma

$$
\begin{aligned}
D(t)^{1/2} \int_{G/M} f(xtx^{-1})\, dx &= D(t)^{1/2} \int_N dn \int_K f(kntn^{-1}k^{-1})\, dk \\
&= D(t)^{1/2} \int_N \overline{f}(ntn^{-1}\, dn \\
&= D(t)^{1/2} \int_N \overline{f}(nmn^{-1}t^{-1} \cdot m)\, dn \\
&= \delta_P^{-1/2}(t) \int_N \overline{f}(nt)\, dn \ .
\end{aligned}
$$

For any unramified character of M let Θ_{π_χ} be the unique function defined on the conjugates of regular elements of M by

$$\Theta_{\pi_\chi}(gtg^{-1}) = \frac{1}{|W|}\, \frac{\sum_W w\chi(t)}{D(t)^{1/2}} \ .$$

and extend it to all of G^{reg} by setting it equal to 0 off the conjugation of M.

Finally:

[charcater] **9.7. Theorem.** *For any function f in $\mathcal{H}(G)$*

$$\operatorname{trace} \pi_\chi(f) = \int_G f(g)\Theta_{\pi_\chi}(g)\, dg$$

Proof. Since $D(t)$ is invariant under W, the two previous Propositions imply that

$$
\begin{aligned}
\int_M \overline{f}_P(t)\chi(t)\, dt &= \int_M \chi(t)|D(t)|^{1/2}\, dt \int_{G/M} f(gtg^{-1})\, dg \\
&= \frac{1}{|W|} \sum_W \int_M w\chi(t)|D(t)|^{1/2}\, dt \int_{G/M} f(gtg^{-1})\, dg \\
&= \int_M \frac{1}{|W|} \sum_W \frac{w\chi(t)}{|D(t)|^{1/2}}\, |D(t)|\, dt \int_{G/M} f(gtg^{-1})\, dg \\
&= \int_G f(g)\Theta_{\pi_\chi}(g)\, dg \ . \ \square
\end{aligned}
$$

10. The Satake transform

I want to emphasize here several features of the Satake isomorphism, and for that reason I'll sketch its proof. In this section let $\mathbb{D} = \mathbb{Q}[q^{\pm 1/2}]$.

[unram] 10.1. Proposition. *For any unramified character χ and w in W, $c_{w\chi} = c_\chi$.*

Proof. From the Corollary to the first Proposition in the previous section, since the normalization factor $D(m)$ is W-invariant. $\square$

In a moment I'll prove that the Satake transform is an isomorphism. The proof can be motivated by a simple example, that of $\mathrm{PGL}_2(k)$. Let

$$\omega = \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix} \text{ modulo the centre .}$$

If $m = 0$ then

$$\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) = c_\chi(\mathfrak{char}_K) = 1$$

and otherwise

$$\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) = q^{m/2} \left[\left(\frac{\omega^{m+1} - \omega^{-(m+1)}}{\omega - \omega^{-1}} \right) - q^{-1} \left(\frac{\omega^{m-1} - \omega^{-(m-1)}}{\omega - \omega^{-1}} \right) \right]$$
$$= Q_m - Q_{m-2}$$

where

$$Q_m = q^{m/2} \left(\omega^m + \omega^{m-2} + \cdots + \omega^{-(m-2)} + \omega^{-m} \right) .$$

Thus in this case $\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) = \delta_P^{-1/2}(\omega^m)\, \omega^m$ modulo terms of lower degree. This is a general phenomenon, and the basis for the proof.

On the lattice $\mathcal{T}$ define $x \leq y$ to mean that (in additive notation)

$$x = y - \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee(\varpi) \quad (n_\alpha \geq 0) .$$

This order on $\mathcal{T}$ induces one on $\mathcal{T}^{--}$ and filtrations on both Hecke algebras $\mathcal{H}(G//K)$ and $\mathcal{H}(M//M(\mathfrak{o}))^W$. For t in $\mathcal{T}^{--}$ define

$$\mathcal{H}(G//K)_t = \left\{ \sum_{u \in \mathcal{T}^{--}, u \leq t} f_u \, \mathfrak{char}_{KuK} \right\}$$
$$\mathcal{H}(M//M(\mathfrak{o}))_t = \left\{ \sum_{u \leq t} f_u u \right\}$$
$$\mathcal{H}(M//M(\mathfrak{o}))_t^W = \mathcal{H}(M//M(\mathfrak{o}))^W \cap \mathcal{H}(M//M(\mathfrak{o}))_t$$

For t in $\mathcal{T}^{--}$ and w in W, $wt \leq t$. This implies that the associated graded ring for both $\mathcal{H}(G//K)$ and $\mathcal{H}(M//M(\mathfrak{o}))^W$ is just $\mathbb{D}[\mathcal{T}^{--}]$, with the ring structure arising from the monoid structure of $\mathcal{T}^{--}$.

[sattrans] **10.2. Proposition.** *The Satake transform is compatible with these filtrations, and induces multiplication by $\delta_P^{-1/2}(t)$ on the t-component of the associated graded module.*

Proof. This follows immediately from the calculation of the Satake transform in terms of $f_P(m)$ and this:

[minus] **10.3. Lemma.** *(a) If m_1 and m_2 are elements of M^{--} and $Km_1K \cap Nm_2 \neq \emptyset$, then $m_2 \leq m_1$; (b) for $m \in M^{--}$ and n in N, $nm \in KmK$ if and only if $n \in N(\mathfrak{o})$.*

This is essentially 2.6.11(3)–(4) in the book of Macdonald when G is simply connected, but the general case follows from that one.

For an example, suppose G to be $\mathrm{GL}_n(k)$. Then G/K can be identified with $\mathfrak{o}$-lattices of rank n in k^n. The group $\mathcal{T}$ may be identified with all diagonal matrices with entries of the form ϖ^m, and $\mathcal{T}^{--}$ is the subset of those with $m_1 \geq m_2 \geq \ldots\ldots \geq m_n$. That

$$G = \bigcup_{t \in \mathcal{T}^{--}} KtK$$

is elementary divisor theory. A matrix g lies in the double coset of

$$t = \varpi^m = \begin{bmatrix} \varpi^{m_1} & & & \\ & \varpi^{m_2} & & \\ & & \ddots & \\ & & & \varpi^{m_n} \end{bmatrix}, \quad \text{where } m = (m_1, \ldots, m_m)$$

with $m_1 \geq \ldots \geq m_n$ if and only if for each r the g.c.d. $\varepsilon_r(g)$ of all $r \times r$ minor determinants of g is that of t, which is

$$\varepsilon_r(t) = \varpi^{m_{n-r+1} + \cdots + m_n} \ .$$

In particular, the integer m_n is the least m with the property that all entries in g are of the form $x\varpi^m$ with x in $\mathfrak{o}$. It can be proven easily by induction that for ν in N

$$\varepsilon_r(\nu\varpi^m) \geq \varepsilon_r(\varpi^m), \quad \varepsilon_n(\nu\varpi^m) = \varepsilon_n(\varpi^m)$$

and that $\varepsilon_r(\nu\varpi^m) = \varepsilon_r(\varpi^m)$ for all r only when ν lies in $N(\mathfrak{o})$.

11. Root data and group structure

In this section I'll summarize in a bit more detail than I have so far the structure of reductive groups. The principal references I am aware of for this section are [Demazure-Grothendieck:1970] and [Springer:1979]. The standard reference for root systems is [Bourbaki:1968].

For the moment, let F be an arbitrary field.

Suppose G to be a connected split reductive group defined over F. A **Borel pair** of G is a pair $\mathfrak{B} = (B, T)$ with B a Borel subgroup of G and T a maximal torus in B. If $\mathfrak{B} = (B, T)$ is a Borel pair, then the adjoint action of T on the nilpotent radical of $\mathfrak{b}$ decomposes into a direct sum of one-dimensional **root spaces**. If $\Sigma_\mathfrak{B}$ is the set of all characters arising in this decomposition, there exists a unique subset $\Delta = \Delta_\mathfrak{B}$ with the property that every λ in $\Sigma_\mathfrak{B}$ can be expressed as $\lambda = \sum_{\alpha \in \Delta} n_\alpha \alpha$ with non-negative integers n_α. It is called the **basis** of $\Sigma_\mathfrak{B}$. The adjoint action of T on all of $\mathfrak{g}$ decomposes into the direct sum of $\mathfrak{t}$ and the one-dimensional root spaces $\mathfrak{g}_\alpha$ where α ranges over $\Sigma_\mathfrak{B}$ and $-\Sigma_\mathfrak{B}$.

Inner automorphisms act transitively on Borel pairs. A Borel subgroup is its own normalizer and the stabilizer of a maximal torus within one is that torus. Therefore if g conjugates $\mathfrak{B}_1 = (B_1, T_1)$ to $\mathfrak{B}_2 = (B_2, T_2)$ the induced isomorphism $X_*(g)$ of $X^*(T_1)$ with $X^*(T_2)$ depends only on the pair and not specifically on g. Similarly for the coweight lattices $X_*(T_i)$. We can therefore define the **canonical root lattice** L_G and its dual $L_G^\vee$ associated to G. An element of this lattice may be identified with the collection of all triples (B, T, λ) where λ lies in $X^*(T)$ and the λ for different Borel pairs correspond under the canonical identification. Since the various $\Sigma_{B,T}$ and $\Delta_{B,T}$ also correspond under the canonical identifications, we may in fact define a canonical quadruple

$$\mathcal{R}_G = (L_G, \Delta_G, L_G^\vee, \Delta_G^\vee) \,.$$

It is usually referred to in English as the **canonical based root datum** of G. (Some linguistic confusion is certainly possible here, since a root datum is an array of data. The use of the singular goes back to [Demazure-Grothendieck:1970], who first defined "une donnée radicielle".) The literature also defines the **canonical root datum** $(L, \Sigma, L^\vee, \Sigma^\vee)$, but in these notes this will not occur. In summary:

[borelpair] **11.1. Proposition.** *For each Borel pair $\mathfrak{B}$ there exists a canonical isomorphism*

$$\kappa_\mathfrak{B} \colon X_*(T) \longrightarrow L_G$$

taking $\Delta^\vee$ to $\Delta_G^\vee$, and likewise for the dual objects. For any inner automorphism γ this diagram commutes:

$$X_*(T) \overset{X_*(\gamma)}{\longrightarrow} X_*(\gamma(T))$$
$$\kappa_\mathfrak{B} \searrow \quad \swarrow \kappa_{\gamma\mathfrak{B}}$$
$$L_G$$

The assignment of $\mathcal{R}_G$ to G is to some extent functorial. We'll not need to know the most general case.

[gamma] 11.2. Corollary. *If γ is any automorphism of G, the automorphism*

$$L_G(\gamma) = \kappa_{\gamma\mathfrak{B}} X_*(\gamma) \kappa_{\mathfrak{B}}^{-1} \colon\ L_G \longrightarrow L_G$$

of L_G depends only on the class of γ modulo inner automorphism.

Or, in other words, the map $\gamma \mapsto L_G(\gamma)$ depends only on the image of γ in the quotient of outer automorphisms

$$\mathrm{Outer}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G) \ .$$

It induces a canonical map from this quotient to the group $\mathrm{Aut}(\mathcal{R}_G)$ of automorphisms of the based root datum.

The relationship between automorphisms of G and automorphisms of $\mathcal{R}_G$ can be made more precise. An **épinglage** $\mathcal{E}$ of G is made up of (1) a Borel pair (B, T) and (2) an array (X_α) of elements of the root spaces $\mathfrak{b}_\alpha$ for $\alpha \in \Delta_{B,T}$. Each X_α determines as well an embedding of the additive group $\mathbb{G}_a$ into B_α. Automorphisms of G transport *épinglages* in an obvious way. In general, it will interchange the elements of the array (X_α) so as to be compatible with the action on $\Delta_{\mathfrak{B}}$. Thus an automorphism takes X_α to X_β where $\beta = L_G^\vee(\gamma^{-1})\alpha$. (There seems to be no good English equivalent for the French word 'épinglage'. The word 'épingler' means 'to pin', and the image that comes to mind most appropriately is that of a mounted butterfly specimen. [Kottwitz:1984] uses 'splitting' for what most call 'épinglage', but this is not compatible with the common use of 'deploiement', the usual French term for 'splitting'.) Ian Macdonald, among others, has suggested that retaining the French word *épinglage* in these notes is a mistake, and that it should be replaced by the usual translation 'pinning.' This criticism is quite reasonable, but I rejected it as leading to non-colloquial English. The words 'pinning' as noun and 'pinned' as adjective are commonly used only to refer to an item of clothing worn by infants, and it just didn't sound right.

The following is the simplest case of one of the principal results (XXIII.4.1) of [Demazure-Grothendieck:1970] (also discussed by [Springer:1979]).

[bpconjugate] 11.3. Proposition. *If $E = (B, T, (X_\alpha))$ and $E' = (B', T', (X'_{\alpha'}))$ are two épinglages then any automorphism of $\mathcal{R}_G$ lifts to a unique automorphism of G taking E to E'.*

If $E' = E$ this leads to:

[canproj] 11.4. Corollary. *The canonical projection from $\mathrm{Outer}(G)$ to $\mathrm{Aut}(\mathcal{R}_G)$ is an isomorphism. Épinglages give rise to splittings of the sequence*

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Outer}(G) \longrightarrow 1 \ .$$

Conversely, we can start with a datum $\mathcal{R} = L, \Delta, L^\vee, \Delta^\vee)$ and construct a group G with $\mathcal{R}_G$ isomorphic to $\mathcal{R}$. The first thing to be decided is what conditions to impose on the datum.

Suppose Δ to be a finite set, embedded as a basis into a lattice L_Δ, which is in turn embedded in a lattice L. Suppose also given a map $\alpha \mapsto \alpha^\vee$ from Δ to a set $\Delta^\vee$ of linearly independent elements in the dual lattice $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$, and that

- for α in Δ, $\langle \alpha, \alpha^\vee \rangle = 2$;
- for $\alpha \neq \beta$ in Δ, $\langle \alpha, \beta^\vee \rangle \leq 0$;
- $\langle \alpha, \beta^\vee \rangle = 0$ if and only if $\langle \beta, \alpha^\vee \rangle = 0$.

These are the conditions that the matrix $(C_{\alpha,\beta})$ with $C_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle$ be an integral **Cartan matrix**. With the first assumption, the linear transformation

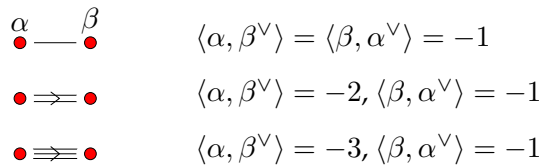$$s_\alpha : v \longmapsto v - \langle v, \alpha^\vee \rangle \alpha$$

is a reflection in L—that is to say it fixes points in the hyperplane $\langle v, \alpha^\vee \rangle = 0$ and takes α to $-\alpha$. So is its contragredient $s_{\alpha^\vee} = s_\alpha^\vee$ in $L^\vee$. These reflections generate a **Coxeter group** in the vector space $L \otimes \mathbb{R}$—i.e. one defined by certain simple generators and relations. It is called the **Weyl group** of the system. In general, it will be the Weyl group of a Kac-Moody algebra, so the last condition on a root datum, at least in this article, is that this Weyl group be finite:

- the s_α generate a finite group.

Associated to the Cartan matrix is the **Dynkin graph** of the matrix, whose nodes are elements of Δ, with an edge from α to a distinct node β if $\langle \alpha, \beta^\vee \rangle \neq 0$. This edge is oriented if

$$n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle < 4 ,$$

and assigned a multiplicity indicated graphically according to the following diagrams:

$$\overset{\alpha}{\bullet} \!\!-\!\!\overset{\beta}{\bullet} \qquad \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$$

$$\bullet \!\Rrightarrow\! \bullet \qquad \langle \alpha, \beta^\vee \rangle = -2, \langle \beta, \alpha^\vee \rangle = -1$$

$$\bullet \!\Rrightarrow\! \bullet \qquad \langle \alpha, \beta^\vee \rangle = -3, \langle \beta, \alpha^\vee \rangle = -1$$

The condition on finiteness is in practice usually verified by checking that each of the connected components of the Dynkin graph belong to one of the known list of graphs of a given rank giving rise to a finite Weyl group, which are given, for example, in [Bourbaki:1968].

In these circumstances, let Σ be the (finite) orbit of Δ under W, $\Sigma^\vee$ that of $\Delta^\vee$. This pair forms a **root system**, for which Δ forms a basis. In particular, if Σ^+ is the subset

of positive roots, those of the form $\sum_\Delta n_\alpha \alpha$ with all the n_α non-negative integers, then

$$\Sigma = \Sigma^+ \cup -\Sigma^+ \ .$$

The complete set $(L, \Delta, L^\vee, \Delta^\vee)$ is a **based root datum**. (In [Grothendieck-Demazure:1970] it is called 'une donnée radicielle épinglée', thus emphasizing usefully the parallel with the term 'groupes épinglée'. I see no way to do this in colloquial English.)

Since the Weyl group is finite, there exists an invariant positive definite inner product $\bullet$ on L. Since the root reflection s_α is an orthogonal reflection,

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\alpha \bullet \alpha} \right)$$

for all α and β in Δ. This implies the matrix equation $2(\alpha \bullet \beta) = DC$ where C is the Cartan matrix and D the diagonal matrix with entries $\alpha \bullet \alpha$. This in turn implies that the Cartan matrix is non-singular.

A root datum is called **semi-simple** if the lattice L_Δ in L spanned by Δ has finite index in L. Since the Cartan matrix is non-singular, this is equivalent to the condition that $(L^\vee)_{\Delta^\vee}$ have finite index in $L^\vee$. In general, let $L^\vee_{\mathrm{der}}$ be the λ in $L^\vee$ such that some non-zero integral multiple $n\lambda$ lies in L_Δ.

[quadruple] 11.5. Proposition. *If $\mathcal{L} = (L, \Delta, L^\vee, \Delta^\vee)$ is a root datum then the quadruple*

$$\mathcal{L}_{\mathrm{der}} = \left((L^\vee_{\mathrm{der}})^\vee, \Delta, L^\vee_{\mathrm{der}}, \Delta^\vee \right)$$

is also one. If $\mathcal{L}$ is the root datum of a reductive group G then $\mathcal{L}_{\mathrm{der}}$ is that of the derived group G_{der}.

Because $L^\vee_{\mathrm{der}}$ is saturated in $L^\vee$, the short exact sequence

$$0 \longrightarrow L^\vee_{\mathrm{der}} \longrightarrow L^\vee \longrightarrow L^\vee / L^\vee_{\mathrm{der}} \longrightarrow 0$$

is a sequence of lattices, which therefore splits (albeit non-canonically). Every root datum is therefore obtained from a semi-simple one by adding on a lattice summand.

If one is given a based root datum $(L, \Delta, L^\vee, \Delta^\vee)$ then, according to a theorem originally due to Chevalley, there exists an essentially unique split reductive group G defined over F giving rise to it. Accounts of this can be found in [Chevalley:1961] and [Demazure-Grothendieck:1970]. The group is determined only up to isomorphism, but the standard proof constructs the group together with an *épinglage*, which rigidifies the construction. The passage from the root datum to the group of rational points is entirely constructive. Results of R. Steinberg, extended somewhat in [Demazure-Grothendieck:1970] and discussed clearly in the recent account [Cohen-Taylor:2002] tell how to describe explicitly the group of rational points by generators and relations. The paper by Cohen et al. goes on to give reasonable algorithms for group operations, representing elements of the group in terms of

the Bruhat decomposition. The algorithms they describe have been implemented in the computer program MAGMA. Slightly more explicitly, if T is the torus whose character group is L and E/F a field extension, its group of E-rational elements may be identified with $L^\vee \otimes E^\times$. The group G is defined as an algebraic group, and $G(E)$ turns out to be the group generated by symbols $s(t)$ for t in $T(E)$ and $e_\alpha(x)$ for α in Δ or $-\Delta$ and x in E. The group $G(E)$ is that generated by the $s(t)$ and $e_\alpha(x)$ subject to certain relations laid out in the paper by [Cohen et al.:2002].

In principle the root datum tells everything about the group. If G_{der} is the derived group of G and T_{der} the intersection of G_{der} with the torus T, the lattice $L^\vee_{\mathrm{der}}$ is the image of $X_*(T_{\mathrm{der}})$ in $L^\vee = X_*(T)$. The group G_{der} will be simply connected precisely when this is equal to $L^\vee_{\Delta^\vee}$, and equal to the adjoint group when it is $(L_\Delta)^\vee$. The quotient $L^\vee/L^\vee_{\mathrm{der}}$ is isomorphic to the co-character group of the torus G/G_{der}. The lattice $L^\vee/L^\vee_{\mathrm{der}}$ is the coroot lattice of the quotient of G by G_{der}.

The centre Z_G of G won't in general be connected, but it will be a multiplicative group, hence determined by its character group $X^*(Z_G)$. The group of its rational points, for example, may be identified with

$$Z_G(F) = \mathrm{Hom}(X^*(Z_G), F^\times) .$$

If $\mathcal{R}_G = (L, \Delta, L^\vee, \Delta^\vee)$ then

$$X^*(Z_G) = L/L_\Delta .$$

12. Root data for quasi-split groups

Suppose E/F to be a Galois extension with group $\mathcal{G}$, and G a connected quasi-split group defined over F that splits over E. Let $\mathfrak{B} = (B, T)$ be a Borel pair defined over F. Any element of $\mathcal{G}$ gives rise to an automorphism of $X^*(T)$ that preserves $\Delta_{\mathfrak{B}}$, hence an automorphism of the root datum $\mathcal{R} = (L, \Delta, L^\vee, \Delta^\vee)$ associated to G over E.

Conversely, a result from the previous section shows that any homomorphism from $\mathcal{G}$ to the $\mathrm{Aut}(\mathcal{R})$ lifts to a homomorphism from $\mathcal{G}$ to $\mathrm{Outer}(G)$, hence by Galois descent to a quasi-split reductive group obtained from the split form of G over F. We have a bijection between homomorphisms φ from $\mathcal{G}$ to the automorphism group of G and quasi-split groups defined over F that are isomorphic over E to the split group G determined by $\mathcal{R}$. If $\gamma \mapsto \varphi_\gamma$ is the homomorphism then the group of F-rational points on G_φ is

$$G_\varphi(F) = \{ g \in G(E) \,|\, \varphi_\gamma(g) = \gamma^{-1}(g) \text{ for all } \gamma \in \mathcal{G} \} \ .$$

In the cases we care about, F will be a $\mathfrak{p}$-adic field and E an unramified extension, therefore $\mathcal{G}$ a cyclic group. In this case, the homomorphism from $\mathcal{G}$ to $\mathrm{Aut}(G)$ is determined by the image of the Frobenius $\mathfrak{F}$.

For example, let E be a quadratic extension of F, $G = \mathrm{SL}_3(F)$, φ_σ the involution

$$X \longmapsto J \,{}^t C^{-1} J$$

of G, where σ is the conjugation of E/F. The corresponding quasi-split group is the special unitary group $SU_3(E/F)$, which becomes isomorphic to $\mathrm{SL}_3(E)$ over E.

Here

$$x_{\alpha_1}(y) = \begin{bmatrix} 1 & y & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

$$x_{\alpha_2}(y) = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{bmatrix}$$

$$x_{\alpha_3}(y) = \begin{bmatrix} 1 & 0 & y \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

The image of conjugation of E/F swaps x_{α_1} and x_{α_2}, hence takes x_{α_3} to $-x_{\alpha_3}$ since

$$[dx_{\alpha_1}, dx_{\alpha_2}] = dx_{\alpha_3} \ .$$

Any automorphism of $\mathcal{L}$ determines one of its Dynkin diagram as well. If the extension E/F is cyclic, the action on this diagram is completely specified by giving

the orbit of each of the nodes. This is usually indicated by **Tits diagrams**. Here is the one for SU_3:

which indicates that conjugation swaps the nodes. Here are all of the Tits diagrams for the absolutely simple groups, along with their conventional designations taken from [Tits:1979]:

$$^2A'_{2n}$$

$$^2A'_{2n-1}$$

$$^2D_n$$

$$^3D_4$$

$$^2E_6$$
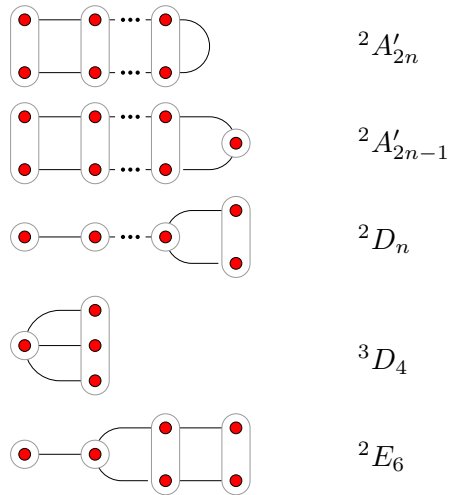
One important thing to notice is that the connected component of each orbit has only one or two nodes in it, and even that it has two only in the case of $^2A'_{2n}$. This can easily be proven directly. This also implies that every simply connected unramified quasi-split group of rational rank one is either $\mathrm{SL}_2(F)$ or $SU_3(E/F)$ where F is some unramified extension of k.

13. The L-group

To get a quick idea of what's coming, let $G = \mathrm{GL}_n$. The unramified characters of $A/A(\mathfrak{o})$ are parametrized by n-tuples $(x_1, \ldots, x_n)$ with each x_i in $\mathbb{C}^\times$. Different n-tuples give rise to the same homomorphism from $\mathcal{H}(G//K)$ to $\mathbb{C}^\times$ if and only if one is a permutation of the other—i.e. the homomorphisms are parametrized by invertible diagonal $n \times n$ matrices up to permutation, or in other words semi-simple conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$. This apparent coincidence is the simplest case— simple enough to be deceptive—of something first pointed out by Langlands. It has proven extraordinarily fruitful in understanding the nature of automorphic forms.

Throughout this section, suppose that G be as usual an unramified, connected, reductive group defined over k.

- *Suppose at first G to be in fact split over k.*

Up to isomorphism, the structure of G is completely determined by its root datum $\mathcal{R} = (L, \Delta, L^\vee, \Delta^\vee)$. The dual $\mathcal{R}^\vee = (L^\vee, \Delta^\vee, L, \Delta)$ of $\mathcal{R}$ is also a root datum. Associated to it is a connected reductive group $\widehat{G}$ defined over $\mathbb{C}$, and in this I choose a Borel pair $(\widehat{B}, \widehat{T})$ where (B, T) is a Borel pair in G. I'll often confound $\widehat{G}$ and $\widehat{G}(\mathbb{C})$.

If $G = \mathrm{GL}_n(k)$, for example, then $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$. If F is semi-simple and simply connected, the $\widehat{G}$ is teh adjoint form of the dual root system. For example, the dual of C_n is B_n, and hence if $G = \mathrm{SL}_n(k)$ then $\widehat{G} = \mathrm{PGL}_n(\mathbb{C})$, and if $G = \mathrm{Sp}(2n)$ then $\widehat{G} = \mathrm{SO}(2n+1)$.

The dual group of $A = T$ is a complex torus $\widehat{A}$. The group of rational homomorphisms

$$\mathrm{Hom}(A/A(\mathfrak{o}), \mathbb{C}^\times) = \mathrm{Hom}(X_*(A), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{A}), \mathbb{C}^\times)$$

is that of unramified characters of A, but it is also by definition the group of points of $\widehat{A}$. Different complex characters give rise to the same homomorphism from $\mathcal{H}(G//K)$ to $\mathbb{C}$ if and only if they are W-conjugates of each other, which is to say if and only if the corresponding points of $\widehat{A}(\mathbb{C})$ lie in the same conjugacy class in $\widehat{G}(\mathbb{C})$. In summary:

- *If G is split, the irreducible unramified representations of G are parametrized by semi-simple conjugacy classes in $\widehat{G}$.*

This will have a remarkable generalization to the case when G is not necessarily split.

Any function in the Hecke algebra $\mathcal{H}(G//K)$ is associated by the Satake transform to a function on the characters of $A/A(\mathfrak{o})$, namely $\chi \mapsto c_\chi(f)$. This is invariant under W, and hence determines a function on the conjugacy class of χ when it is

identified with an element of $\widehat{A}$ (still assuming G to be split). What functions arise in this way? Each t in $\mathcal{T}^{--}$ corresponds to a weight of the complex torus $\widehat{T}$. Its Weyl orbit forms the extremal weights of a unique irreducible representation ρ_t of $\widehat{G}$ whose highest weight is among these. Let $\chi(\rho_t)$ be its character.

As is already implicit in the proof of the Satake isomorphism, every function in the **integral Hecke algebra** made up of $\mathbb{Z}$-linear combinations of the $\mathfrak{char}_{KtK}$ is mapped under the Satake transform to a $\mathbb{Z}$-linear combination of conjugation invariant functions $\delta_P^{-1/2}(t)\,\chi(\rho_t)$, and conversely every such linear combination lies in the image of the Satake transform.

- *If G is split, the integral Hecke algebra $\mathcal{H}(G//K)$ is isomorphic to the ring of conjugation-invariant functions on $\widehat{G}$ generated by the functions $\delta_P^{-1/2}(t)\,\chi(\rho_t)$ as t ranges over $\mathcal{T}^{--}$.*

Let me point out one interesting place where $\widehat{G}$ illuminates the theory of spherical functions. Recall that for $G = \mathrm{PGL}_2(k)$

$$\mathfrak{S}(\mathfrak{char}_{K\omega^m K}) = \begin{cases} 1 & m = 0 \\ Q_1 & m = 1 \\ Q_m - Q_{m-2} & m \geq 2 \end{cases}$$

where

$$Q_m = q^{m/2}\left(\omega^m + \omega^{m-2} + \cdots + \omega^{-(m-2)} + \omega^{-m}\right).$$

The element ω is a generator of $A/A(\mathfrak{o})$, which may be identified with a generator of $X^*(\widehat{A}(\mathbb{C}))$. In terms of this identification, Q_m is equal to $q^{m/2}$ times the character of the irreducible representation of $\widehat{G} = SL_2(\mathbb{C})$ of dimension $m+1$ and highest weight ω^m. For general groups we expect to meet the Weyl character formula in similar circumstances.

- *Now I drop the assumption that G be split.*

Recall that Γ is the Galois group of k_{nr}/k. In this case, the isomorphism class of G is determined by its root datum $\mathcal{R} = (L, \Delta, L^\vee, \Delta^\vee)$ together with a homomorphism φ from Γ to $\mathrm{Aut}(\mathcal{R})$, hence determined by the image of $\mathfrak{F}$. As before, we define $\widehat{G}$ be a connected reductive group defined over $\mathbb{C}$ together with an isomorphism of its root datum with $\mathcal{R}^\vee$.

If we are given an *épinglage*, it determines a lifting from $\mathrm{Aut}(\mathcal{R}^\vee)$ to $\mathrm{Aut}(\widehat{G})$, hence a homomorphism from Γ to $\mathrm{Aut}(\widehat{G})$. Since the image fixes an *épinglage*, it is what [Kottiwtz:1984] calls an **L-action**.

Following [Kottwitz:1984], I say that a **dual group** for G is a reductive group $\widehat{G}$ over the complex numbers $\mathbb{C}$ together with an L-action of Γ and a Γ-isomorphism of the based root datum of $\widehat{G}$ with $\mathcal{R}^\vee$. The corresponding **unramified L-group** ${}^L G$ of G is the semi-firect product of W_{nr}, the cyclic group generated by the Frobenius, and $\widehat{G}$.

- *In rest of this section, because the role of the Levi component of P is predominantly as a maximal torus, I shall write it as T, not M.*

When G is split, the L-group is just the direct product of W_{nr} and $\widehat{G}$, and the Frobenius in the L-group plays no important role. For arbitrary unramified groups, one has to take it into account. The way to parametrize unramified representations in general is based on a remarkable observation of Langlands:

[lcon]] 13.1. Theorem. *Every semi-simple ^LG conjugacy class in $\widehat{G} \times \mathfrak{F}$ contains an element $\tilde{t} \times \mathfrak{F}$ with $\tilde{t}$ in $\widehat{T}$. The image of $\tilde{t}$ in $\widehat{A}/W$ depends only on the original conjugacy class, and induces a bijection between semi-simple conjugacy classes in $\widehat{G} \times \mathfrak{F}$ and $\widehat{A}/W$.*

There are few places in the literature where this is proven, and indeed the L-group of an arbitrary unramified group seems to be something of a neglected animal, suffering by comparison with the simpler split groups. But, amazingly, the arbitrary case was dealt with by Langlands right from the beginning. The original construction of the L-group is in the letter [Langlands:1967] to Weil that introduced it along with several conjectures about automorphic forms. Other accounts are in [Langlands:1971] and [Borel:1978]. In the survey [Casselman:1998] I sketched a proof of this Proposition that incorporated a suggestion of Kottwitz, but it was perhaps a bit too condensed as well as full of typographical errors to be completely satisfactory, and I'll expand it here. Most of what I say is taken from Borel's Corvallis exposition, but even there can be found a few minor errors.

The proof starts with a simple calculation.

[sameas] 13.2. Lemma. *In $\widehat{G} \times \mathfrak{F} \subseteq {}^LG$, $\widehat{G}$-conjugacy is the same as ^LG-conjugacy.*

Proof. If g and x lie in $\widehat{G}$ then

$$ g \cdot x \times \mathfrak{F} \cdot g^{-1} = gxg^{-\mathfrak{F}} \times \mathfrak{F} \, . $$

On the other hand

$$ \mathfrak{F} \cdot x \times \mathfrak{F} \cdot \mathfrak{F}^{-1} = \mathfrak{F}x\mathfrak{F}^{-1} \times \mathfrak{F} = x^{\mathfrak{F}} \times \mathfrak{F} = x^{-1}xx^{\mathfrak{F}} \times \mathfrak{F} $$
$$ = x^{-1} \cdot x \times \mathfrak{F} \cdot x \, . \; \square $$

The equivalence $x \sim gxg^{-\mathfrak{F}}$ is called **twisted conjugacy**. So what this result says is that $g_1 \times \mathfrak{F}$ and $g_2 \times \mathfrak{F}$ are conjugate under either $\widehat{G}$ or ^LG if and only if g_1 and g_2 are twisted conjugates.

In order to make Langlands' Theorem at least plausible, let's look first at the case when G is the torus T itself. The injection of $X_*(A)$ into $X_*(T)$ induces a surjection

$$ \widehat{T} = \mathrm{Hom}(X_*(T), \mathbb{C}^\times) \longmapsto \widehat{A} = \mathrm{Hom}(X_*(A), \mathbb{C}^\times) \, . $$

The following result asserts precisely that the fibres of this projection are the twisted conjugacy classes in $\widehat{T}$.

[trivialimage] 13.3. Lemma. *An element $\widehat{t}$ of $\widehat{T}$ has trivial image in $\widehat{A}$ if and only if it is of the form $uu^{-\mathfrak{F}}$ for some u in $\widehat{T}$.*

Proof. The image of $X_*(A)$ in $X_*(T)$ is the submodule of Galois invariants. The quotient in the short exact sequence

$$0 \longrightarrow X_*(A) \longrightarrow X_*(T) \longrightarrow \mathcal{L} = X_*(T)/X_*(A) \longrightarrow 0$$

therefore has no torsion, and the short exact sequence therefore splits. Thus the kernel of the projection from $\widehat{T}$ to $\widehat{A}$ is a connected torus $\mathrm{Hom}(\mathcal{L}, \mathbb{C}^\times)$. This kernel contains the image of $u \mapsto uu^{-\mathfrak{F}}$. The Lie algebra of $\widehat{T}$ decomposes into a direct sum of the $\mathfrak{F}$-invariants and the image of $\mathfrak{F} - I$. The first is isomorphic to the Lie algebra of $\widehat{A}$ and the second to the Lie algebra of the kernel. Since $\mathfrak{F} - I$ is invertible on its image, the Lemma follows, since exponentiation maps a Lie algebra onto its torus. ◻

Step 1. *Every semi-simple conjugacy class in $\widehat{G} \times \mathfrak{F}$ contains an element $\widehat{t} \times \mathfrak{F}$ with $\widehat{t}$ in $\widehat{T}$.*

Suppose $x \times \mathfrak{F}$ to be a semi-simple element of $\widehat{G} \times \mathfrak{F}$. We want to find g in $\widehat{G}$ such that

$$g \cdot x \times \mathfrak{F} \cdot g^{-1} = gxg^{-\mathfrak{F}} \times \mathfrak{F} = y \times \mathfrak{F}$$

with y in $\widehat{T}$. Equivalently, we want to find g such that $y = gxg^{-\mathfrak{F}}$ satisfies

$$y\widehat{T}y^{-1} = \widehat{T}, \quad y\widehat{B}y^{-1} = \widehat{B} .$$

because the first places y in the normalizer of $\widehat{T}$, and the second then places it in $\widehat{T}$ itself.

Now if $H = \widehat{T}$ or $\widehat{B}$, then since $H^{\mathfrak{F}} = H$, $yHy^{-1} = H$ means that

$$\begin{aligned} yHy^{-1} &= gxg^{-\mathfrak{F}} \cdot H \cdot g^{\mathfrak{F}}x^{-1}g^{-1} = H \\ g^{-1}Hg &= x \cdot (g^{-1}Hg)^{\mathfrak{F}} \cdot x^{-1} \\ &= (x \times \mathfrak{F})\, g^{-1}Hg\, (x \times \mathfrak{F})^{-1} \end{aligned}$$

In other words, we are looking for g such that gHg^{-1} is fixed under conjugation by $x \times \mathfrak{F}$. Theorem 7.5 of [Steinberg:1968] asserts that there exists a Borel subgroup B_* and a maximal torus T_* contained in B_* fixed by $x \times \mathfrak{F}$, and the pair (B_*, T_*) will be conjugate to $(\widehat{B}, \widehat{T})$. (Some define a semi-simple element of a possibly disconnected group to be one which stabilizes a pair $T \subset B$.) (This argument is taken from [Kottwitz-Shelstad:1999].) ◻

Step 2. For this next step, let W_{nr} be the Weyl group of G with respect to T, over k_{nr}. The **restricted Weyl group** W is, according to a very general result about reductive groups, the quotient of the stabilizer in W_{nr} by its subgroup of elements that act trivially on $X_*(A)$. But since $X_*(A)$ is the fixed submodule of the Galois group in $X_*(T)$ and the Galois group stabilizes the fundamental chamber, $X_*(A)$ contains points in the interior of the fundamental chamber. Hence any w in the stabilizer of A acting trivially on it must be itself trivial. Thus W may be identified with the stabilizer of $X_*(A)$ in W_{nr}. The k-roots of G are the restrictions to A of the roots Σ_{nr}, and correspond bijectively to the orbits of the Frobenius on Σ_{nr}. The restrictions of the basic roots are the restrictions of a basis for Σ_{nr}^+. The elements of W may also be identified with the elements of W_{nr} fixed by the Galois group. Better than that:

[restrictedweyl] 13.4. Lemma. *Every element of the restricted Weyl group can be represented by an element of $N_{\widehat{G}}(\widehat{T})$ fixed by the Frobenius automorphism.*

I follow the argument of [Borel:1978]. According to the proof of Theorem 5.3 of [Borel-Tits:1965], the restricted Weyl group is generated by reflections corresponding to the roots in the basis Δ_{nr}. More precisely, if α lies in Δ_{nr}, let $A_{\{\alpha\}}$ be the kernel of α in A. The restricted reflection is the unique element s of order two in W_{nr} fixing $X_*(A)$ and trivial on $X_*(A_{\{\alpha\}})$. It suffices to prove the claim for such an s, say corresponding to the orbit D in Δ_{nr}.

By the construction of the group $\widehat{G}$ from the root datum, we can find a family of elements $e_{\pm\alpha}$ in $\mathfrak{g}_{\pm\alpha}$ for each α in Δ, permuted among each other by $\mathfrak{F}$. For each α the element $[e_\alpha, e_{-\alpha}] = h_\alpha \neq 0$ lies in $\widehat{\mathfrak{t}}$, and $h_{\mathfrak{F}\alpha} = \mathfrak{F}h_\alpha$. The elements

$$h = \sum_{\alpha \in D} h_\alpha, \quad e_+ = \sum_{\alpha \in D} e_\alpha, \quad e_- = \sum_{\alpha \in D} e_{-\alpha}$$

are all fixed by $\mathfrak{F}$, and since the sum of a positive and a negative root is never a root

$$[e_+, e_-] = h$$

and

$$[h, e_+] = \sum_{\alpha, \beta \in D} \langle \alpha, \beta^\vee \rangle e_\beta \ .$$

The number

$$d_\beta = \sum_\alpha \langle \alpha, \beta^\vee \rangle$$

is independent of β. The number of roots in any connected component of D can be only 1 or 2, since the Galois group acts transitively on it, and the component must be the Dynkin graph of type A_1 or A_2 (as I have already pointed out in the previous section), so that d_β must in fact be 1 or 2. Thus h, $e_\pm$ span a three dimensional Lie algebra fixed element-wise by $\mathfrak{F}$, whose corresponding three-dimensional subgroup in $\widehat{G}$ is also fixed point-wise by $\mathfrak{F}$. Pick in it an element w representing the non-trivial Weyl element. □

Step 3. It follows from Step 1 that the map

$$\widehat{T} \times \mathfrak{F} \longrightarrow \widehat{G} \times \mathfrak{F}/\mathrm{Int}(\widehat{G})$$

is a surjection. If $\mathcal{N}$ is defined to be the inverse image of W with respect to the projection from $N_{\widehat{G}}(\widehat{T})$ to W_{nr}, it follows from Step 2 that every element of $\mathcal{N}$ can be written as a product of w with $w^{\mathfrak{F}} = w$ and $\widehat{t}$ in $\widehat{T}$. If $n = \widehat{u}w$ then

$$n \cdot \widehat{t} \times \mathfrak{F} \cdot n^{-1} = \widehat{u} \cdot w\widehat{t}w^{-1} \cdot \widehat{u}^{-\mathfrak{F}} \times \mathfrak{F}$$

so that $\mathcal{N}$ conjugates $\widehat{T} \times \mathfrak{F}$ to itself.

[inclusion] 13.5. Proposition. *The map*

$$\widehat{T} \times \mathfrak{F}/\mathcal{N} \longrightarrow \widehat{G} \times \mathfrak{F}/\mathrm{Int}(\widehat{G})$$

induced by inclusion is an injection as well as a surjection.

Proof. The following argument is attributed in [Borel:1978] to T. Springer. Suppose that

$$gt_1g^{-\mathfrak{F}} = t_2 \ .$$

Suppose g to have the Bruhat factorization $g = n_1\widehat{t}wn_2$ with w fixed by $\mathfrak{F}$. Thus

$$n_1\widehat{t}wn_2 \cdot t_1 \cdot n_2^{-\mathfrak{F}}w^{-1}\widehat{t}^{-\mathfrak{F}}n_2^{-\mathfrak{F}} = t_2$$
$$n_1 \cdot \widehat{t}wt_1w^{-1} \cdot w \cdot t_1^{-1}n_2t_1^{-1} = t_2n_1^{\mathfrak{F}} \cdot \widehat{t}^{\mathfrak{F}} \cdot w \cdot n_2^{\mathfrak{F}}$$
$$= t_2n_1^{\mathfrak{F}}t_2^{-1} \cdot t_2\widehat{t}^{\mathfrak{F}} \cdot w \cdot n_2^{\mathfrak{F}}$$

which implies that

$$t_2\widehat{t}^{\mathfrak{F}} = \widehat{t}wt_1w^{-1}, \quad t_2 = \widehat{t}w \cdot t_1 \cdot w^{-1}\widehat{t}^{-\mathfrak{F}} \ . \ \square$$

Step 4. Langlands' Theorem now follows from:

[yyy] 13.6. Proposition. *Projection from $\widehat{T}$ to $\widehat{A}$ is equivariant with respect to the projection from $\mathcal{N}$ to W. The induced map*

$$\widehat{T} \times \mathfrak{F}/\mathcal{N} \longrightarrow \widehat{A}/W$$

is a bijection.

Proof. Surjectivity follows from the surjectivity of the projection from $\widehat{T}(\mathbb{C})$ to $\widehat{A}(\mathbb{C})$. Equivariance and injectivity both follow immediately from the fact that any w in W has a representative in $N_{\widehat{G}}(\widehat{T})$ which is fixed by $\mathfrak{F}$. $\square$

This concludes the proof of Langlands' Theorem. In combination with Satake's Theorem, it says that the unramified principal series are parametrized by semi-simple conjugacy classes in the coset $\widehat{G} \times \mathfrak{F}$ of ^{L}G, the inverse image in ^{L}G of the Frobenius automorphism in W_{nr}. Another way to phrase this is to say that they are parametrized by semi-simple splittings of the canonical projection from ^{L}G to W_{nr}. Or, equivalently, they are parametrized by diagrams

$$
\begin{array}{ccc}
W_{nr} & \longrightarrow & ^{L}G \\
& \searrow \quad \swarrow & \\
& W_{\mathrm{nr}} &
\end{array}
$$

where the arrow from W_{nr} to itself is the identity. Homomorphisms from W_{nr} to ^{L}G fitting into this diagram are called L-**homomorphisms**.

- *From now on, if π is an irreducible unramified representation of G, let $\mathfrak{F}_{\pi}$ be the corresponding conjugacy class in $\widehat{G} \times \mathfrak{F}$.*

The most immediate gain from introducing the L-group is the association of L functions to automorphic forms. If π is an unramified representation of G and ρ a finite dimensional representation of ^{L}G then Langlands defines

$$
L(s, \pi, \rho) = \frac{1}{\det\left(I - \rho(\mathfrak{F}_{\pi})q^{-s}\right)} \ .
$$

Of course this is well defined since it depends only on the conjugacy class of $\mathfrak{F}_{\pi}$. Since a unitary representation has to have bounded matriz coefficients, Macdonald's formula implies a bound on s that guarantees convergence for this function in a right half-plane. It is precisely these L functions which should play the role in automorphic forms that the Artin L-functions do in number theory.