

Archimedes' calculation of π

Archimedes found explicit bounds on the value of π by a method that remained the principal technique for over a thousand years. It depends on approximating the area of a circle by the area of inscribed and circumscribed regular polygons of many sides.

1. The modern formula

If we inscribe in the unit circle a regular polygon of n sides, its area is equal to

$$A_n = n \cdot (1/2) \cos(\pi/n) \cdot 2 \sin(\pi/n) = (n/2) \sin 2\pi/n = \text{(say)} (n/2)s_n$$

where

$$s_n = \sin 2\pi/n.$$

Archimedes used this formula only implicitly, and used explicitly a formula to go from n sides to $2n$ sides, starting with $n = 4$. where the calculation is very simple. In modern terms

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

which for θ in the range we are concerned with is also equal to

$$\sin 2\theta = 2 \sin \theta \sqrt{1 - \sin^2 \theta}.$$

We want to invert this, and find a formula for $\sin \theta$ in terms of $\sin 2\theta$. Squaring both sides above we get

$$\sin^2 2\theta = 4 \sin^2 \theta (1 - \sin^2 \theta), \quad 4 \sin^4 \theta - 4 \sin^2 \theta + \sin^2 2\theta$$

leading to

$$\sin^2 \theta = \frac{4 \pm \sqrt{16 - 16 \sin^2 2\theta}}{8} = \frac{1 \pm \sqrt{1 - \sin^2 2\theta}}{2}$$

Which sign do we choose here? If θ is small then $\sin 2\theta$ and $\sin \theta$ will both be small, and this dictates the choice of the minus sign. Thus

$$\sin^2 \theta = \frac{1 - \sqrt{1 - \sin^2 2\theta}}{2}$$

This formula, for reasons I don't want to explain here, is awkward, but it can be rewritten in a more useful manner. Just multiply top and bottom by $1 + \sqrt{1 - \sin^2 2\theta}$. This gives us

$$\sin^2 \theta = \frac{\sin^2 2\theta}{2(1 + \sqrt{1 - \sin^2 2\theta})}, \quad \sin \theta = \frac{\sin 2\theta}{\sqrt{2(1 + \sqrt{1 - \sin^2 2\theta})}}.$$

One virtue of this version is that it can be checked. If θ is small the $\sin \theta$ is about the same as θ , and this formula does indeed imply that for θ small $\sin \theta$ is about half of $\sin 2\theta$.

Thus Archimedes' method for calculating π starts with

$$\begin{aligned} A_n &= (n/2)s_n \\ s_4 &= \sin \pi/2 \\ &= 1 \\ A_4 &= (4/2) \cdot 1 \\ &= 2 \\ s_{2n} &= \frac{s_n}{\sqrt{2(1 + \sqrt{1 - s_n^2})}}. \end{aligned}$$

For example

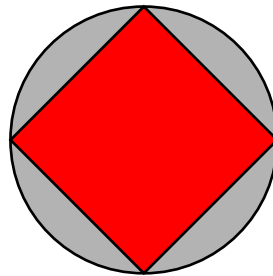
$$\begin{aligned}
 s_8 &= \frac{s_4}{\sqrt{2(1 + \sqrt{1 - s_4^2})}} \\
 &= \frac{1}{\sqrt{2}} \\
 A_8 &= (8/2)s_8 \\
 &= \frac{4}{\sqrt{2}} \\
 &\sim 2.83
 \end{aligned}$$

Exercise 1. Calculate A_{16} , A_{32} by Archimedes' formula.

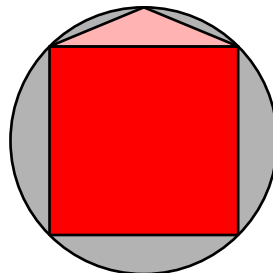
2. Accuracy

Archimedes' method is reasonably practical. It becomes more so if we have ahead of time some idea of how large we have to take n to get the accuracy we want, say to 10 decimals. This reduces to two questions; (1) How accurate is A_4 as an approximation of π ? (2) What happens to the accuracy when we double n ?

(1) The error in A_4 is the outer shaded area in this figure:



The point here and in the next step also is that to a good approximation one of these four regions looks like part of the area inside a parabola determined in another essay by Archimedes. The area of the region is therefore approximately $4/3$ that of the triangle that can be fit inside:



The area of this triangle is one half base times height, or $(1/2)\sqrt{2}(1 - 1/\sqrt{2})$. The error in A_4 is therefore about

$$\frac{4}{3} \frac{1}{2} \sqrt{2} (1 - 1/\sqrt{2}) \sim 1.10.$$

Not a bad estimate, compared to the true value of about 1.14.

(2) The proof of Archimedes' theorem about the area of the circle shows that the error is better than halved at each step. In fact, though, the same estimate in terms of parabolas, which becomes better and better as n gets large, shows that the error is pretty much divided in 4 each step.

3. References

Archimedes, *Measurement of the circle*.