Mathematics 309 — Mid-term solutions

1. *A lens consists of a half a sphere of radius R, with the flat side on the back, perpendicular to'the lens axis. A ray leaves a point at a distance $3R$ in front of the lens at an angle of $\pi/36$ with respect to the axis, passes through the lens, then hits the axis again somewhere on the far side. The lens has a refraction index of 1.5. (a) Where would it hit the axis on the far side according to the linear approximation? (b) Exactly where does the ray hit?*

(a) Let $T = \pi/36$. If we assume that the initial point is on the x-axis and the point we are looking for is $(f, 0)$ then with $k = 3$

$$\begin{aligned}\begin{bmatrix} 1 & f \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & R/n \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -(n-1)/R & 1 \end{bmatrix}\begin{bmatrix} 1 & kR \\ & 1 \end{bmatrix}\begin{bmatrix} 0 \\ T \end{bmatrix} &= \begin{bmatrix} 1 & f \\ & 1 \end{bmatrix}\begin{bmatrix} 1/n & (k+1)R/n \\ -(n-1)/R & 1-k(n-1) \end{bmatrix}\begin{bmatrix} 0 \\ T \end{bmatrix} \\ &= \begin{bmatrix} 1 & f \\ & 1 \end{bmatrix}\begin{bmatrix} T(k+1)R/n \\ T\big(1-k(n-1)\big) \end{bmatrix}\end{aligned}$$

which leads to

$$f = \frac{(k+1)R/n}{k(n-1)-1} = (16/3)R\,.$$

If you don't assume that the initial point is on the axis, you start at say $(-4, y)$ instead. In this case the intercept will depend on y, and explicitly in a similar way you find that

$$f = \frac{y/n + T(k+1)R/n}{T\big(k(n-1)-1\big) + y(n-1)/R}\,.$$

Under the assumption that $y = 0$ there is another way to do it. The point where the linear ray hits is the same as the conjugate plane since the x-axis is also a ray. Now if a lens has transfer matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and one plane sits ℓ to the left, the conjugate plane on which images from the first focus is at distance f where

$$\begin{bmatrix} 1 & f \\ & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & \ell \\ & 1 \end{bmatrix}$$

has upper right entry equal to 0. This implies that

$$f = -\frac{A\ell + B}{C\ell + D}\,.$$

In our case the transfer matrix is

$$\begin{bmatrix} 1/n & R/n \\ -(n-1)/R & 1 \end{bmatrix}$$

leading again to

$$f = \frac{(k+1)R/n}{k(n-1)-1} = (16/3)R\,.$$

(b) Suppose the center of the lens is at $(0, 0)$. May as well set $R = 1$. Suppose a ray at position $P_0 = (x_0, 0)$ leaves at angle t_0, or direction $v_0 = [\cos t_0, \sin t_0]$. It hits at the point $(x_0, 0) + s[\cos t_0, \sin t_0]$ lying on the left half of the sphere. Then

$$\begin{aligned}(x_0 + s\cos t_0)^2 + s^2\sin^2 t_0 &= 1 \\ x_0^2 + 2x_0 s\cos t_0 + s^2 &= 1 \\ s^2 + (2x_0\cos t_0)s + (x_0^2 - 1) &= 0 \\ s &= -x_0\cos t_0 \pm \sqrt{1 - x_0^2\sin^2 t_0}\end{aligned}$$

(How would you do a quick check?) In our case, x_0 is negative, and we choose the $-$ sign. Let $P_1 = (x_0 + s\cos t_0, s\sin t_0)$. Let A be the angle such that $P_1 = (\cos A, \sin A)$, and let $B = \pi - A$, so also $P_1 = (-\cos B, \sin B)$.

The normal vector to the sphere at P_1 is also $[\cos A, \sin A]$. The angle of incidence is $i_1 = t_0 + B$. If

$$r_1 = \arcsin\left(\frac{\sin i_1}{n}\right)$$

then the new ray direction is $v_1 = [\cos t_1, \sin t_1]$ where $t_1 = t_0 - (i_1 - r_1)$. The new ray starts at P_1 in direction v_1. Explicitly

$$P_1 = (-0.9641, 0.2656)$$
$$v_1 = [0.9994, -0.0343]$$

It hits the back face at

$$P_2 = P_1 + sv_1 = (x_1 + s\cos t_1, y_1 + s\sin t_1)$$

where $x_2 = x_1 + s\cos t_1 = 0$. So

$$s_1 = -\frac{x_1}{\cos t_1}, \quad y_2 = y_1 + s\sin t_1 = y_1 - x_1\tan t_1 \ .$$

The new ray has incidence angle $i_2 = t_1$ and refracted angle

$$r_2 = \arcsin(n\sin i_2)$$

so the new direction is $t_2 = t_1 - (i_2 - r_2)$.

$$P_2 = (0.0000, 0.2326)$$
$$v_2 = [0.9987, -0.0514]$$

This new ray hits the x-axis at

$$P_3 = P_2 + s[\cos t_2, \sin t_2]$$

where

$$y_2 + s\sin t_2 = 0, \quad s = -\frac{y_2}{\sin t_2} \ .$$

Then

$$x_3 = x_2 + s\cos t_2 = s\cos t_2 = -\frac{y_2}{\tan t_2} \ .$$

$$P_3 = (4.5173, 0)$$

2. *A wave with a frequency of 2 cycles per second is traveling in 2D at a velocity of 1 unit per second, in the direction of the vector $[-1, 2]$. At time $t = 0$ its crest passes through the origin. (a) Where is the location of that crest at $t = 1$? (b) How far behind it is the next crest? (c) At what time does the crest just behind it pass through the origin?*

(a) In one second the wave translates by $[-1, 2]/\sqrt{5}$. So the wave crest after one second is the line perpendicular to $[-1, 2]$ through the point $(-1, 2)/\sqrt{5}$. Its equation is $-x + 2y = \sqrt{5}$.

(b) In one second, two cycles pass by. So one crest is 1/2 unit behind the previous crest.

(c) $t = 1/2$.

3. *A quadruple rainbow is one associated to rays that reflect 4 times inside a raindrop. Where in the sky might you look for one at sunset?*

For a rainbow of k internal reflections, the turning angle is $\tau = 2(i - r) + k(180 - 2r)$, where

$$r = \arcsin\left(\frac{\sin i}{n}\right)$$

the derivative of this respect to $y = \sin i$ is

$$\frac{2}{\sqrt{1 - y^2}} - \frac{2(k+1)/n}{\sqrt{1 - (y/n)^2}} .$$

This is 0 when

$$y = \pm\sqrt{\frac{(k+1)^2 - n^2}{(k+1)^2 - 1}} .$$

Whe $n = 1.33$ and $k = 4$, we get

$$\begin{aligned}
y &= \pm\sqrt{\frac{5^2 - 1.33^2}{5^2 - 1}} \\
&= 0.9839 \\
i &= \arcsin(0.9839) \\
&= 79.69^\circ \\
r &= 47.71^\circ \\
\tau &= 2(i - r) + k(180 - 2r) \\
&= 2i + 180k - 2r(k+1) \\
&= 402.26^\circ \\
&= 42.26^\circ
\end{aligned}$$

The turning angle is about 42°, which means that you would look towards the Sun to see it.