Chapter 5. Stereographic projection

It is impossible to render paths on a sphere onto a flat surface in such a way that all distances remain the same. In drawing a map of a sphere, therefore, some compromises must be made. Most maps adopt one of two possible strategies: (1) areas are preserved, or (2) angles are preserved. **Stereographic projection** is one way of making maps, and it preserves angles. It has been used since ancient times for this purpose, and its basic geometrical properties were known even then.

1. The projection

The basic idea is very simple. We apply ordinary perspective to a sphere embedded in three dimensions in a particular way.

Recall the formula for perspective projection onto the plane $z = -1$ with the origin as viewing point:

$$x' = \frac{x}{-z}, \quad y' = \frac{y}{-z}, \quad z' = -1 .$$

This maps (x, y, z) onto the point on the plane $z = -1$ lying on the line through (x, y, z) and the origin.

We now restrict this map to the sphere of unit radius, centred at $(0, 0, -1)$. The plane $z = -1$ bisects this sphere into two hemispheres. The origin itself is a point of this sphere, but its image under this map is not defined. If we choose points on the sphere close to the origin then z will be close to 0, and the image of (x, y, z) will be very far away. In effect, the origin is mapped to ∞.

We can also define this map in terms that do not depend so explicitly on a coordinate system. Choose a point on the sphere, which we call the **pole** P of the projection. Construct a plane through the origin whose normal vector is the ray from the origin to P.
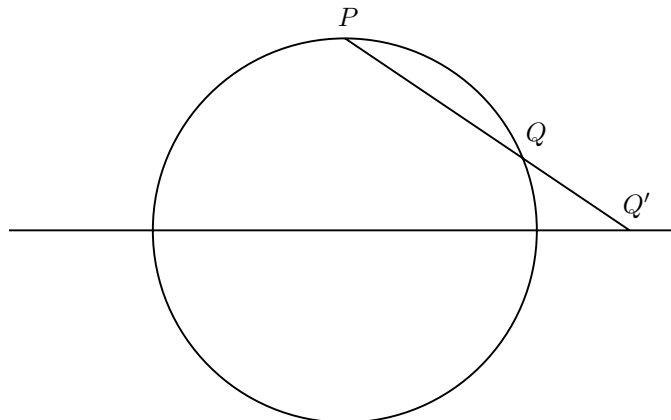
The projection maps all points on the sphere except the pole to a point on this plane. It does this in the following manner: Let Q be a point on the sphere other than P. The line through P and Q will intersect the plane in eactly one point Q', which is the one we map Q to.

The advantage of this construction is that we can now apply it to any sphere. Suppose we take the sphere of radius 1 centred at the origin, for example. Let $P = (0, 0, -1)$ (the south pole, perhaps). The plane of projection

is $z = 0$. If $Q = (x, y, z)$ then the parametrized line through P and Q is $P + t(Q - P) = (tx, ty, 1 - t - tz)$. This intersects $z = 0$ when $t = 1/(1 + z)$, which makes

$$Q' = (x', y', 0), \quad x' = \frac{x}{1 + z}, \quad y' = \frac{y}{1 + z} \ .$$

Exercise 1.1. *What does the north pole get mapped to? What is the image of the 'latitude' $z = c$? The longitude defined by the intersection of the plane $y = cx$ with the sphere? Include sample drawings to illustrate.*

From a technical standpoint, stereographic projection is distinguished by these two properties:

- It preserves the angle between paths.
- It takes circles on the sphere into circles on the plane, except that it takes circles through P into lines.

We shall prove these two properties in the next few sections. First we must recall some simple geometry of circles, and then prove an elementary property of cones.

2. Circles

Suppose we are given a circle and four points located as in the following picture: A and B are opposite ends of a diameter, P is an arbitrary point on the circle and Q is the point on the diameter AB obtained by dropping a perpendicular from P.
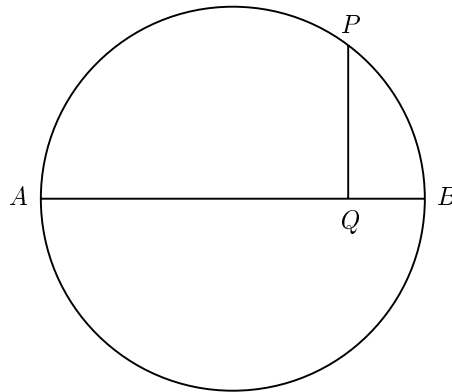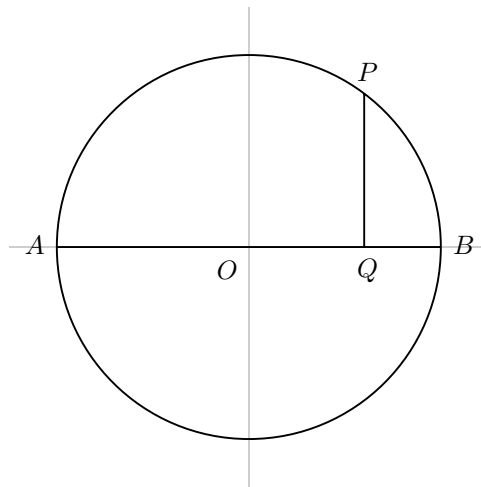

Then we have the following identity:

$$PQ^2 = AQ \cdot QB \ .$$

This is just another way of writing the familiar equation of a circle expressing Pythagoras' Theorem. If we let O be the centre of the circle, r its radius, and let x and y be suitable coordinates then we have

$$PQ = y, \quad AQ = r + x, \quad QB = r - x, \quad AQ \cdot QB = (r + x)(r - x) = r^2 - x^2 = y^2 = PQ^2 \ .$$

The converse is equally simple. If A and B are arbitrary points in the plane then the set of all points P such that $PQ^2 = AQ \cdot QB$, where Q is the foot of the perpendicular from P to the line AB, will be a circle.

The equation is of some historical interest. The Greeks did not use coordinate systems, and this identity was their substitute for the familiar equation of the circle that we use.
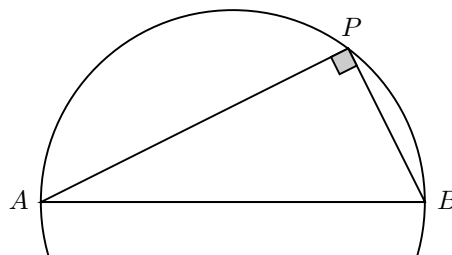
The equation above can be rewritten as

$$\frac{PQ}{AQ} = \frac{QB}{PQ} \ .$$

In this form, it expresses the similarity of the two triangles in the figure below.
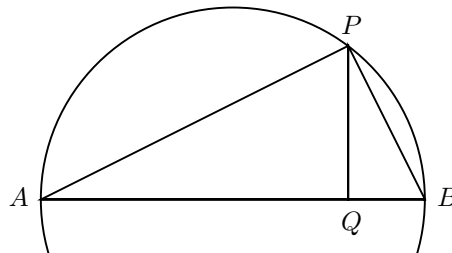
This follows from the well known result that the angle at P in the following figure is a right angle.
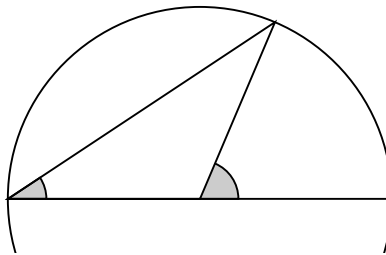

The proof of this I leave as an exercise, as also this result:

- In the following figure, the shaded angle depends only on the chord of the circle opposite to it, as long as that vertex remains on one side of the chord.
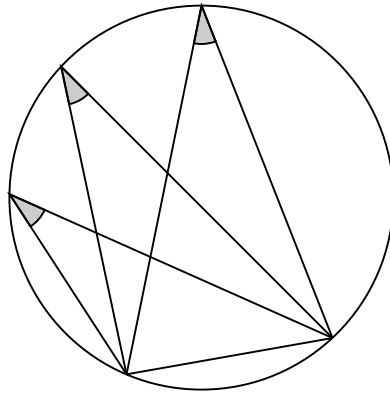
In proving this, you will want to recall the relation between the two angles in this figure:


3. Circular sections of a cone

Place a circle in a plane, and choose a point C not on that plane. Construct the cone with C as vertex intersecting the plane in the given circle. It is simple to see that any section of the cone parallel to the given plane is also a circle. If the cone is **oblique**, that is to say if its axis is not perpendicular to the plane, then there exist other circular sections of the cone as well. In fact there exists a single parallel family of them.

They can be constructed in a relatively simple fashion. Construct a plane perpendicular to the original one, containing the axis of the cone. This plane will contain the vertex, and will also contain a diameter ab of the original circle.
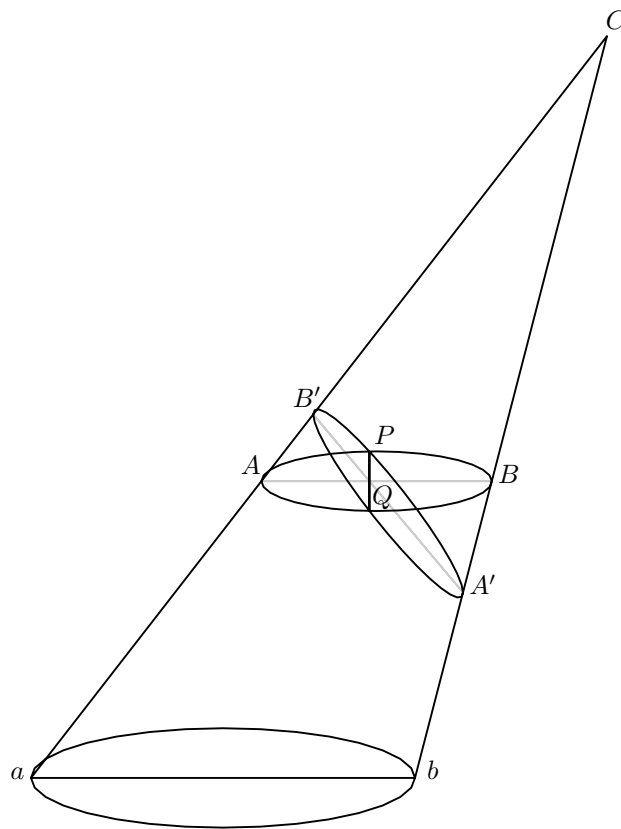
Now construct a plane in the following manner: it is perpendicular to the axial plane, and its intersection with the axial plane cuts off a triangle $A'CB'$ similar to the triangle abC, but with the contrary orientation. We want to show that the section of the cone by this plane is a circle. Let A' and B' be the points of intersection of the axial plane, this contrary plane, and the cone. If P is any point of this intersection and Q is the foot of the perpendicular from P to $A'B'$, we must show that $PQ^2 = A'Q \cdot QB'$. Pass a plane parallel to the original one pass through the line PQ, and let A and B be the points corresponding to a and b. Since this section by this new plane is a circle, we know that $PQ^2 = AQ \cdot QB$. But the following figure explains why $AQ \cdot QB = A'Q \cdot Q'B$.

This result is one of the first results in Apollonius' book on conics, and presumably one of the earliest non-trivial results known to the Greeks about conic sections.
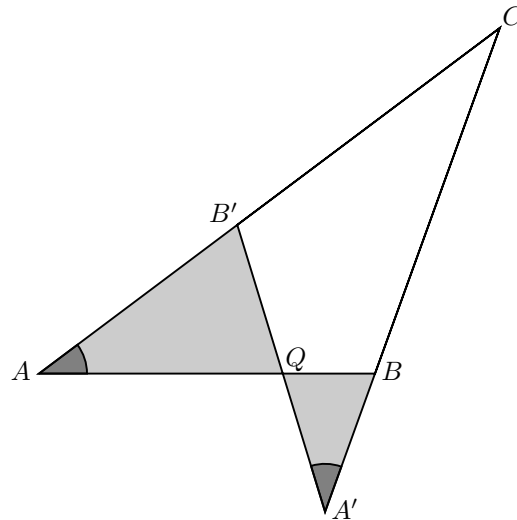
4. Stereographic projection preserves angles

5. Stereographic projection maps circles to circles

The segment AB represents a circular section of the sphere, and $A'B'$ represents its image under stereographic projection. The following figure explains why the section $A'B'$ is a contrary section of the cone generated by P and the circle AB, hence a circle.
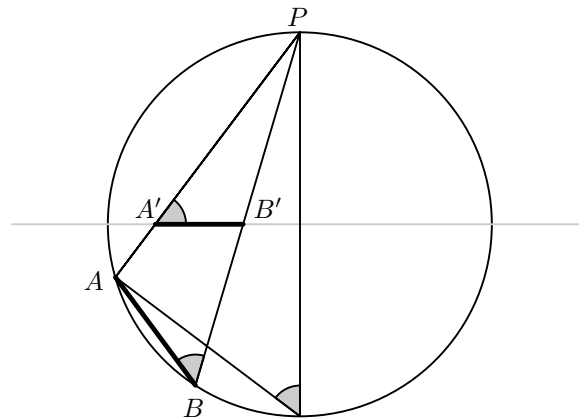
6. Astrolabes

Astrolabes were the first widely used portable computers. They were probably invented and used by the Hellenistic Greeks, perhaps as early as the year 0 of the Christian era. They were used for navigation, astronomical observations, and time-keeping. The first models still in existence are Arabian in origin, from sometime in the 10th century, and were in common use until telescopes superseded them. One of the earliest manuals on the use of an astrolabe was written by the English poet Geoffrey Chaucer, ostensibly for the use of his son. The ones we know about are brass disks averaging about 8 inches across, with a map of the sky and celestial coordinates on one side, and some marks on the other enabling one to take sightings of celestial objects. They are beautiful and intriguing objects to all who have seen them. The construction of astrolabes depends on stereographic projection.