Mathematics 308 — Geometry

Chapter 2. Elementary coordinate geometry

Using a computer to produce pictures requires translating geometry to numbers, which is carried out through a coordinate system. Through nearly all of this course, the coordinate systems we use will have the property that the x and y axes are perpendicular to each other and measured in the same units.

The first topic will recall how to calculate lengths in such a coordinate system, which relies simply on Pythagoras' Theorem. I shall recall in the text a single proof of this, and suggest others in exercises. Of course many proofs are known—the one I present is a variant of Euclid's. It requires a preliminary discussion of shears.

1. Shears

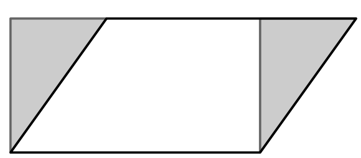
A **shear** is a transformation of a $2D$ figure that has this effect:

It is a bit hard to describe in plain language. Its effect can perhaps be best realized by thinking of the rectangle as a side view of a deck of thick cards:

In other words it slides the components of a figure past each other, and it slides things further if they are higher. From this picture it should be at least intuitively clear that

- *Shears preserve area.*

Roughly speaking this is because sliding a very thin piece of a figure doesn't change its shape. The actual proof that a shear doesn't change area is also very simple, at least if the shear has small enough effect:



The idea is that we lop off a triangle from one end and shift it around to the other in order to make a parallelogram into a rectangle. The reason this works is because we can shift that triangle without distorting it. If the shear is a large one, then it can be expressed as a sequence of small ones applied one after the other, and hence still preserves area. This reasoning also shows that area = base × height.

Of course we have to appeal to some more fundamental result to justify this argument. A rigourous proof can be put together by discussing angles cut off by parallel lines. Ultimately it derives from Euclid's parallel postulate, but I won't discuss this further.

Exercise 1.1. *Read Euclid's proof of his proposition I.35. Reproduce diagrams illustrating his proof in PostScript.*

2. Lengths

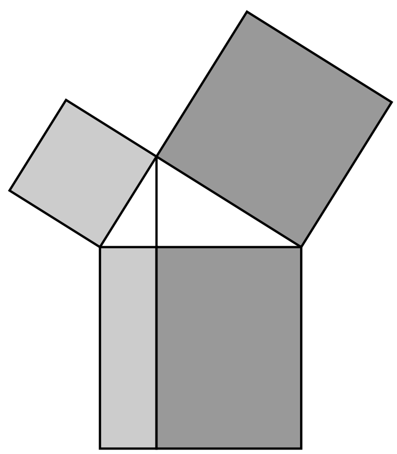
We begin with the statement of Pythagoras' Theorem:

- *For a right triangle with short sides a and b and long side c we have $c^2 = a^2 + b^2$.*

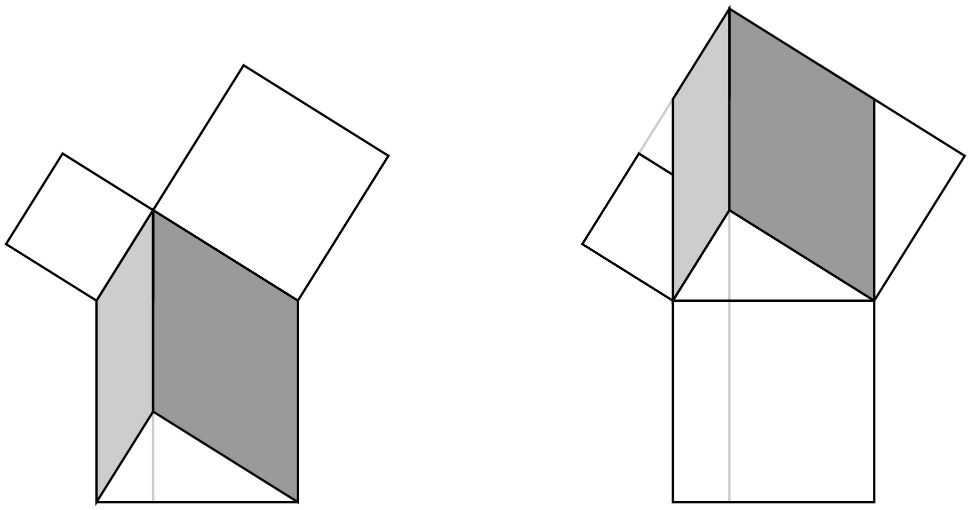
For the coordinate systems we are working with, those in in which x and y are measured uniformly and the x and y axes are perpendicular to each other, this has as consequence that

- *The distance from the origin to (x, y) is $\sqrt{x^2 + y^2}$.*

The point of our proof (and Euclid's) is that one can explicitly decompose the large square into two rectangles, each of which matches one of the smaller squares in area. We do this by dropping a perpendicular from the right angle vertex across to the hypotenuse and through to the base of the large square.



The proof now proceeds by performing a series of shears and translations, which are area-preserving, to transform the rectangles into the corresponding squares.



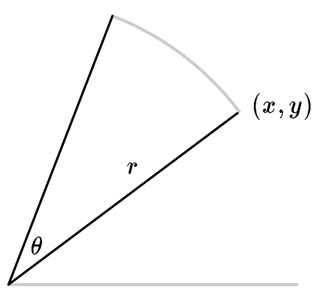
3. Rotations

Suppose we rotate the point in the plane with coordinates (x, y) through an angle of θ. What are the coordinates of the point we then get? The answer is

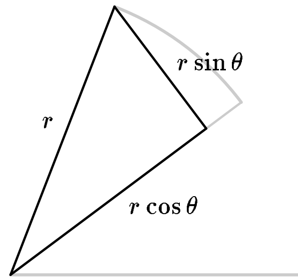
- *If we rotate the point (x, y) around the origin through angle θ, the point we get is*

$$(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \; .$$

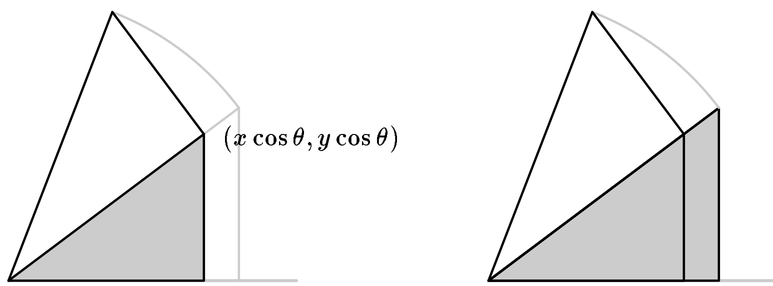
This is a formula used repeatedly, in various guises and in many different circumstances throughout these notes. The proof given here is very direct.



The first step is to drop a perpendicular from the rotated point onto the radius vector of (x, y).

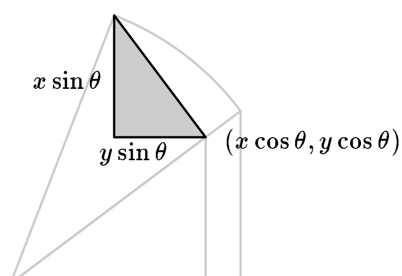


What are the coordinates of the bottom of the perpendicular? Because we have rotated through an angle of θ and rotation preserves distances from the origin, the distance from the origin to the bottom of the perpendicular is $r\cos\theta$, where $r = \sqrt{x^2+y^2}$ is the length of the original vector (x, y). The length of the perpendicular itself is $r\sin\theta$.



Since the triangle on the left is obtained from the one on the right by a simple scaling operation they are similar. The ratio of the long sides is $r\cos\theta : r$, so the coordinates of the bottom of the perpendicular are $(x\cos\theta, y\cos\theta)$.

We now add a triangle to the picture:



It also is similar to the one of the triangles in the previous figure (the angle at its lower right is obtained by a simple rotation from one of the angles in the smaller of those two), and since its long side is $r\sin\theta$ its bottom has length $x\sin\theta$ and the left side length $y\sin\theta$.

But this tells us immediately that the x-coordinate of the rotated point is $x\cos\theta - y\sin\theta$, and its y-coordinate is $y\cos\theta + x\sin\theta$.

If we take (x, y) to be the point $(\cos\varphi, \sin\varphi)$ we get by rotating $(1, 0)$ through an angle of φ, then on the one hand we get the vector $(\cos(\varphi+\theta), \sin(\varphi+\theta))$ that we would get by rotating $(1, 0)$ through an angle of $\varphi+\theta$, and on the other the formula we have just proven gives a different expression. Therefore

$$(\cos(\varphi+\theta), \sin(\varphi+\theta)) = (\cos\varphi\cos\theta - \sin\varphi\sin\theta, \cos\varphi\sin\theta + \sin\varphi\cos\theta)$$

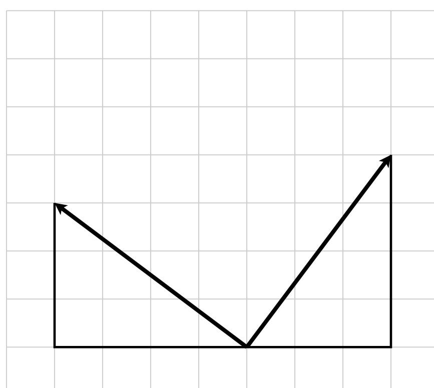
In fact, the rotation formula is equivalent to the pair of trigonometrical formulas

$$\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$$
$$\sin(\varphi+\theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$$

There is one simple case of the rotation formula which is used very often.

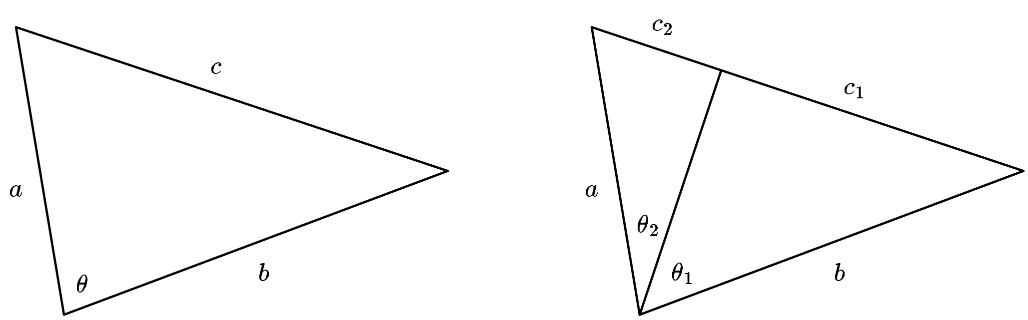
- *If (x, y) are the coordinates of a vector then $(-y, x)$ are the coordinates of the vector rotated through a right angle in the positive direction.*

This can be seen directly:



4. Angles

The cosine sum formula has as an immediate consequence the cosine rule for triangles, which is a generalization of Pythagoras' Theorem.



Let the side opposite the origin have length c. By 'dropping' a perpendicular from the origin onto this side we decompose it into two pieces of length, say, c_1 and c_2. Thus

$$c^2 = (c_1 + c_2)^2 = c_1^2 + c_2^2 + 2c_1c_2$$

On the other hand the original angle θ is decomposed into two parts θ_1, θ_2. We know that

$$\cos\theta = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

Finally, let y be the length of the perpendicular. By Pythagoras' Theorem applied to each of the small triangles and trigonometry

$$\begin{aligned} c_1^2 &= a^2 - y^2 \\ c_2^2 &= b^2 - y^2 \\ c_1 &= a\sin\theta_1 \\ y &= a\cos\theta_1 \\ c_2 &= a\sin\theta_2 \\ y &= a\cos\theta_2 \end{aligned}$$

so that

$$\begin{aligned} c^2 &= (a^2 - y^2) + (b^2 - y^2) + 2ab\sin\theta_1\sin\theta_2 \\ &= a^2 + b^2 - 2ab\ \cos\theta_1\cos\theta_2 + 2ab\sin\theta_1\sin\theta_2 \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

The cosine rule in turn relates angles to dot products.

The dot product of two vectors in n dimensions is the sum of the products of their coordinates:

$$(x_1, x_2, \ldots, x_n) \bullet (y_1, y_2, \ldots, y_n) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\ .$$

There are a number of simple formal algebraic rules it satisfies:

$$\begin{aligned} cx \bullet y &= c(x \bullet y) \\ x \bullet cy &= c(x \bullet y) \\ (x + y) \bullet z &= x \bullet z + y \bullet z \\ x \bullet x &= \|x\|^2 \end{aligned}$$

where $\|x\|$ is the length of the vector x, the distance of its head from its tail.

- *For vectors u and v*

$$u \bullet v = \|u\|\,\|v\|\ \cos\theta$$

where θ is the angle between u and v.

If u and v are vectors, then they form two sides of a triangle. On the one hand, the square of the length of the third side is

$$\|u - v\|^2 = \|u - v\| \bullet \|u - v\| = \|u\|^2 + \|v\|^2 - 2u \bullet v$$

and on the other, by the cosine rule, it is

$$\|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos\theta$$

so by comparison

$$u \bullet v = \|u\|\,\|v\|\cos\theta$$

In particular:

- *The dot product of two vectors is 0 precisely when they are perpendicular to each other.*

5. Lines

There are several ways to determine lines in the plane.

- *A line is determined by a pair of distinct points P and Q.*
- *A line is determined by its equation*

$$Ax + By = C\ .$$

  As a special case of this we have the slope-intercept form

$$y = mx + b$$

  which describes lines which are not dead vertical.
- *A line is determined by a point P and a direction away from that point. If $v = (\Delta x, \Delta y)$ is a vector in that direction then the line is the set of points*

$$P + tv$$

  where t ranges over all real numbers. This is called the **parametric representation** *of the line.*

In this section we shall answer two questions: (1) What is the geometrical significance of the equation

$$Ax + By = C?$$

and (2) If we are given two lines in parametric form, how do we calculate their intersection?

The equation

$$Ax + By = C$$

can be rewritten as

$$(A, B) \bullet (x, y) = C\ .$$

which means that the corresponding line is that of all vectors (x, y) have a fixed dot product with (A, B). If (x_0, y_0) and (x_1, y_1) are two points on the same line then

$$(A, B) \bullet (x_0, y_0) = (A, B) \bullet (x_0, y_0), \qquad (A, B) \bullet (x_0 - y_1, y_0 - y_1) = 0$$

which means that their difference is is perpendicular to (A, B).

There will be exactly one vector (x, y) on the line which is a multiple of (A, B), say $t(A, B)$. We can solve:

$$(A, B) \bullet (tA, tB) = C$$
$$t = \frac{C}{A^2 + B^2}$$

The sign of t will be the sign of C. The length of $t(A, B)$ will be

$$\left| \frac{C}{\sqrt{A^2 + B^2}} \right| \ .$$

In summary:

- *The vector (A, B) is perpendicular to the line $Ax + By = C$.*
- *The signed distance from the origin to this line is*

$$\frac{C}{\sqrt{A^2 + B^2}}\ .$$

The intersection of two lines in parametric form

$$\{P + tu\}, \quad \{Q + tv\}$$

is a point R which satisfies

$$R = P + tu = Q + sv$$

for some numbers t, s. Thus if w is any vector perpendicular to v

$$(P - Q) + tu = sv$$
$$(P - Q + tu) \bullet w = 0$$
$$t = \frac{(Q - P) \bullet w}{u \bullet w}$$

Exercise 5.1. *Given a line $Ax + By = C$ and a point $P = (x_0, y_0)$, find a formula for the perpendicular projection of P onto the line.*