# A Mathematical Approach To Solving Rubik's Cube

by Raymond Tran, UBC Math308 – Fall 2005

History:

"We turn the Cube and it twists us." --Erno Rubik

The Rubiks Cube is a cube consisting of 6 sides with 9 individual pieces on each. The main objective when using one is to recreate it's original position, a solid color for each side, with out removing any piece from the cube. Though it is colorful and looks like a children's toy, there have been many championships for it's completion. It amused five-year-olds yet inspired mathematicians.
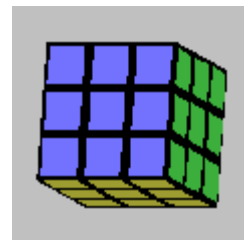
It's unique design was made by an engineer named Erno Rubik, a socialist bureaucrat who lived in Budapest, Hungary. He built the simple toy in his mother's apartment and did not know of the 500 million people who were going to become overly perplexed over it. His first idea of the cube came in the Spring of 1974.

What inspired Erno was the popular puzzle before his called the 15 Puzzle. Invented in the late 1870's, this puzzle consisted of 15 consecutively numbered, flat squares that can be slid around inside a square frame. Sam Loyd created this two dimensional version of the Rubiks Cube. The puzzle was originally called the Magic Cube, or Buvuos Kocka in Hungarian. It was later renamed in honor of it's creator to the Rubiks Cube. Many different cube variations have been made, but the one discussed here is called the standard 3x3x3. It contains 26 little blocks of plastic.

The Rubiks Cube has been a successful product for many years. Though created without great intentions, people have spent millions of dollars on it. Math classes to this day study the complexity of the Cube. Erno, the creator of the cube, became an overly rich man from his ingenious creation.

Introduction:

The cube can rotate around it's center in any way possible, no pieces are restricted to any singular movement. The cube is not easily solved because it does not have a definite scrambled point. This means that there is only one completed situation, where all the sides have one color each. If the cube is anything but that, it is considered scrambled. For example when the cube is complete and one simple rotation is made it is scrambled even though it would be easy to undo that. The cube has 43,252,003,274,489,856,000 (43 quintillion) possible positions, and only one is the correct one. It has been calculated that if every person on earth randomly twisted a cube once every second, about once every three centuries one cube would return to its original state.
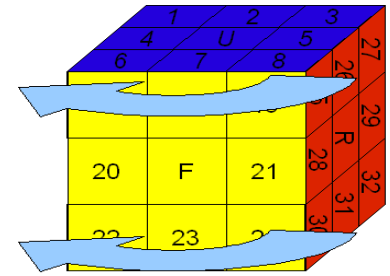
The cube has even been used in college math classes dealing with group theory, a branch of algebra having to do with geometric symmetry developedin the nineteenth century. Group theory shows that a 60 degree rotation of a six-pointed snow flake makes the flakes appearance unchanged. Each group theory is symmetrical, and the cube represents this is after rotation.   The cube can be solved in two ways. One can use sequences to solve piece by piece, or you can attempt to solve it backwards. This means that after the cube is completed and mixed, you can figure what turns were made to mix it and undo them. Mathematicians have tried to find the shortest method of unscrambling, which became known as God's algorithm. God's algorithm relies on a tree structure of all possible scrambled possitions, where a node is a position found by making a move to scramble the cube from a previous node. The root of the tree is the single initial position where the cube is solved. The algorithm searches for the matching scrambled position from the root of the tree and a solution is found by traversing the actions leading to the path found. Although God's algorithm is fast, it is more of a computing approach rather than mathematical approach. Here, we'll attempt to illustrate a simplified version of the mathematical approach by Professor W.D. Joyner of the U.S. Naval Academy.

Notation:

- *X* = turn a face X 90° clockwise (Ex: Uclk is illustrated by the turning the top row of the rubik on the right via direction indicated by the top arrow.)

- *X⁻¹* = turn a face X 90° counterclockwise (Ex: Dclk is illustrated by the turning the bottom row of the rubik on the right via direction indicated by the bottom arrow.)
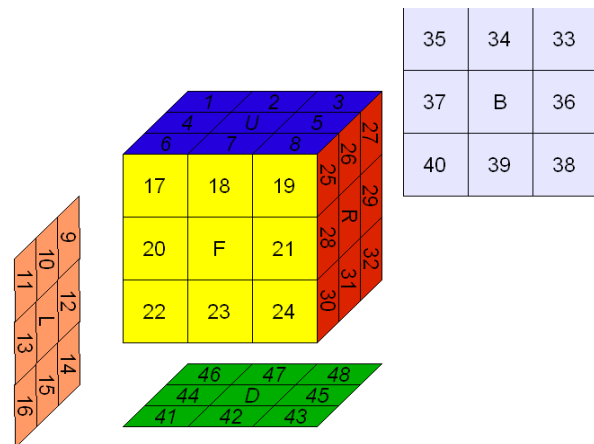
- *X*Y* = sequence X,Y in such order.

Strategy:

First, we label cube as accroding to their faces as follow:

Flat representation of initial cube   *3D representation of initial cube*

From here, the group is generated by the following generators, corresponding to the six faces of the cube:

```
U:= ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19),
L:= ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35),
F:= (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11),
R:= (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24),
B:= (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27),
D:= (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40).
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The size of the group generated by these permutations is 43252003274489856000. It should be noted, however, that the center facet of each face labeled U, L, F, R, B, D does not move like the other facets. The following denotes the mathematical notations used in the solving method:

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Let xʸ=y⁻¹*x*y denote conjugation and [x,y]=x⁻¹*y⁻¹*x*y denote the commutator, for x,y group
elements. If x,y,z denote 3 group elements, let [x,y,z]=x⁻¹*y⁻¹*z⁻¹*x*y*z. If x is a group
element and n>0 is an integer then xⁿ=x*x*...*x (n times).
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The rubik should be solved in 3 stages or levels as follow:

Level 1: Solve the Upper face and the facets around its edges totaling 21 facets.
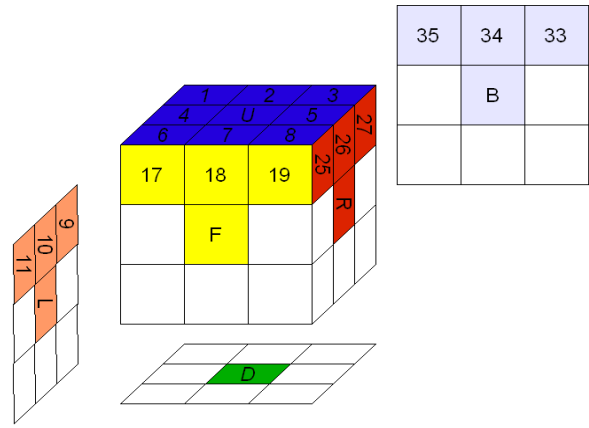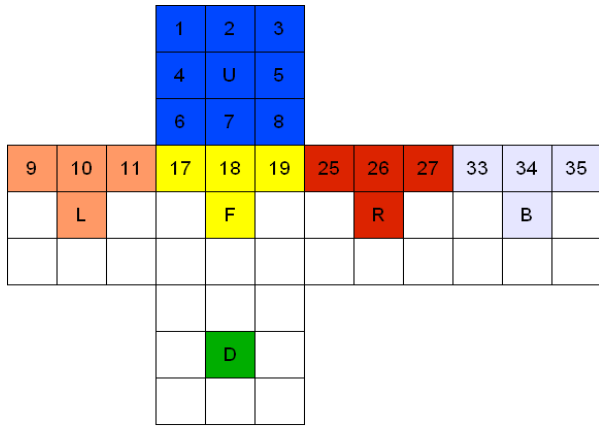
White facets denotes those that are not of concern at this stage

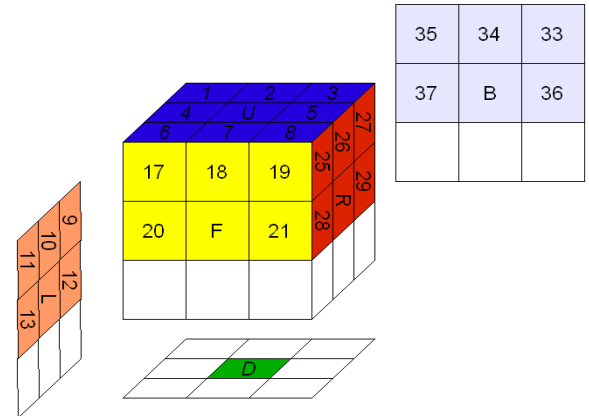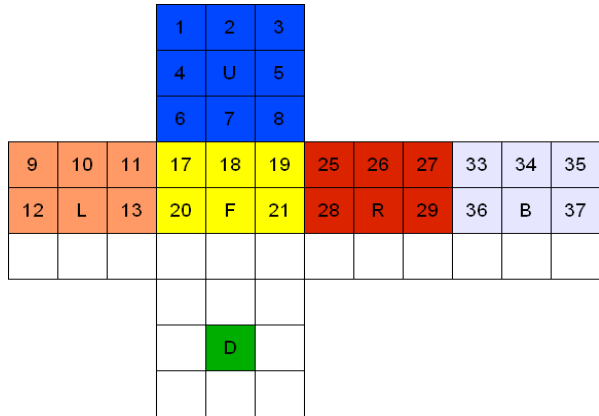To solve level 1, we have the following moves:

1.1) "monotwist": $[F, R^{-1}]^2 = (F^{-1}*R*F*R^{-1})^2$

1.2) "monoswap": $\overset{F^{-1}}{D} * \overset{F}{D^2} * \overset{F^{-1}}{D^{-1}} = (F*D*F^{-1})*(F^{-1}*D*D*F)*(F*D^{-1}*F^{-1})$

1.3) "monoflip": $(\epsilon\ R)^4$, where epsilon (ϵ) is the counterclockwise middle slice quarter turn. Since turning the middle slice 90° is the same as turning the top and bottom slices -90°, this can be rehence can be rewriten as $(U^{-1}*D^{-1}*R)^4$

1.4) "edgeswap": $\overset{(R^2*L^2)}{U^2} = (L^{-1}*L^{-1}*R^{-1}*R^{-1})*(U*U)*(R*R*L*L)$

Level 2: Solve the middle band of 12 facets while leaving results of level 1 in tact.

To solve level 2, we have the following "clean edge" moves:

2.1) $\mathbf{R^2*U*F*B^{-1}*R^2*F^{-1}*B*U*R^2}$ is the top edge 3-cycle (uf,ub,ur)

2.2) $[U, F^{-1}, R]*[U^{-1}, B, R^{-1}]^L$ This flips, but does not permute, the top edges uf, ub
= $\mathbf{(U^{-1}*F*R^{-1}*U*F^{-1}*R)*L^{-1}*(U*B^{-1}*R*U^{-1}*B*R^{-1})*L}$

2.3) $(R^2*U^2)^3$ permutes 2 pairs or edges (uf,ub)(fr,br)
= $\mathbf{(R*R*U*U)^3}$

2.4) $(L^2*F^2*B^2*R^2*F^2*B^2)\char94(D*B^2*F^2)$ permutes 2 pairs of top edges (uf,ul)(ur,ub)
= $\mathbf{(F^{-1}*F^{-1}*B^{-1}*B^{-1}*D^{-1})*(L*L*F*F*B*B*R*R*F*F*B*B)*(D*B*B*F*F)}$

Level 3: Solve the Down face and the facets around its edges totaling 21 facets while leaving results from levels 1 and 2 intact.
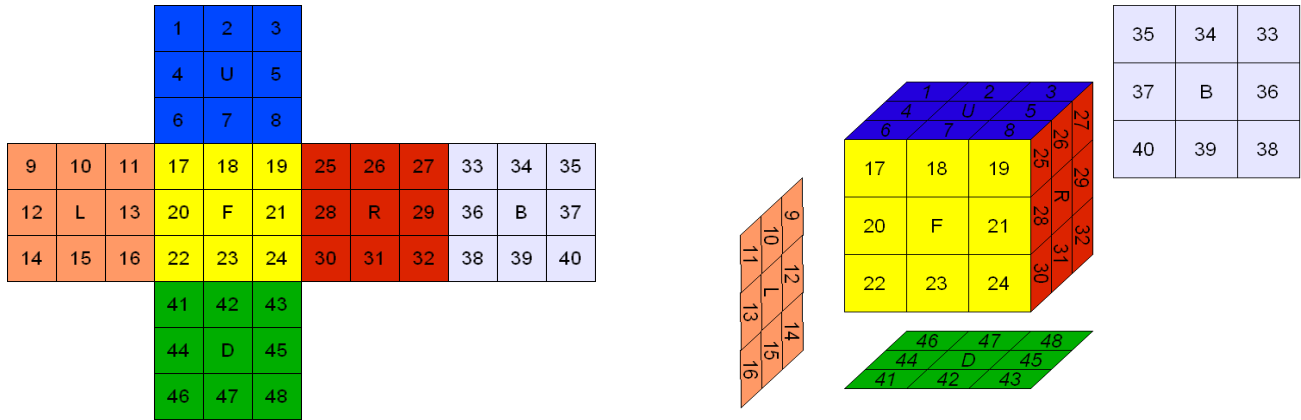
To solve level 3, we have the following "clean corner" and "clean corner-edge" moves:

3.1a) $((D^2)^R*(U^2)^B)^2$ twists the ufr corner clockwise and the bld corner counterclockwise
= $((R^{-1}*D*D*R)*(B^{-1}*U*U*B))^2$

3.1b) The move $(((U^2(D^2)^{\wedge}(F*R^{-1}))^2)^{\wedge}(R^{-1})$ has the same 2-corner-twist effect as the one above.
= $R*(U*U)*(R*F^{-1})*(D*D)*(F*R^{-1})*(U*U)*(R*F^{-1})*(D*D)*(F*R^{-1})*R^{-1}$

3.2) $((D^2)^{\wedge}(F*D^{-1}*R)*U^2)^2$ permutes 2 pairs of corners (ufr,ufl)(ubr,ubl)
= $((R^{-1}*D*F^{-1})*(D*D)*(F*D^{-1}*R)*U*U)^2$

3.3) $[(D^{-1})^R,U^{-1}]$ corner 3-cycle (bru,blu,brd)
= $(R^{-1}*D*R)*U*(R^{-1}*D^{-1}*R)*U^{-1}$

3.4) $B^{\wedge}(U^{-1}*F)*U^2*U^B*U^2*B^{-1}$ permutes two top edges and 2 top corners (ulb,urb)(ub,ur)
= $(F^{-1}*U*B*U^{-1}*F)*U*U*B^{-1}*U*B*U*U*B^{-1}$

A Simpler Method:

Based on the above method, there is a simpler version which works on the same 3-stage principle, but with simpler move sequences. We will now illustrate this method and use to solve a scrambled rubik as an example.

The scrambled cube is created from the initial position using the following moves:
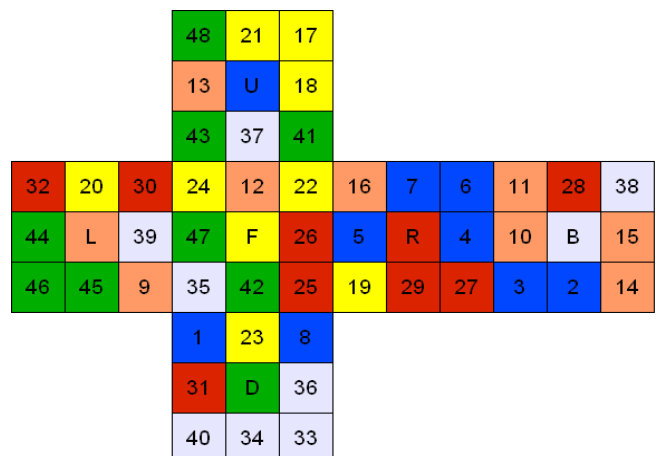U*L*D*R*F*B^{-1}*U^{-1}L^{-1} (ie. Uclk, Lclk, Dclk, Rclk, Fclk, Bcnt, Ucnt, Lcnt). The resulting rubik is shown on the right.
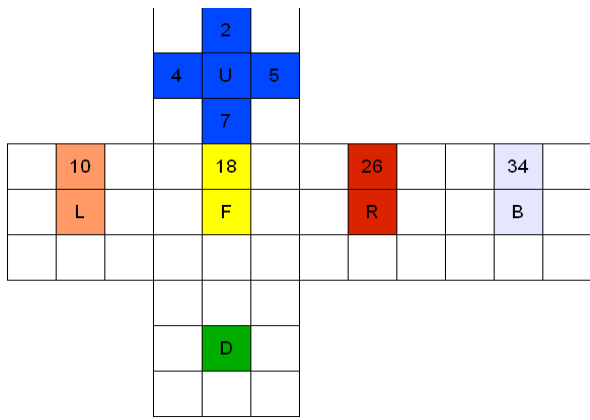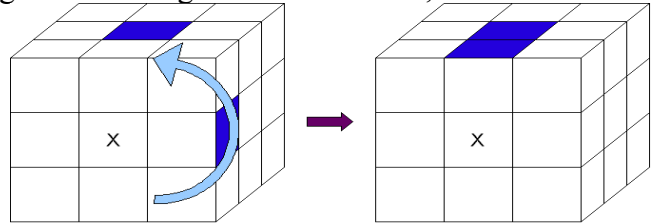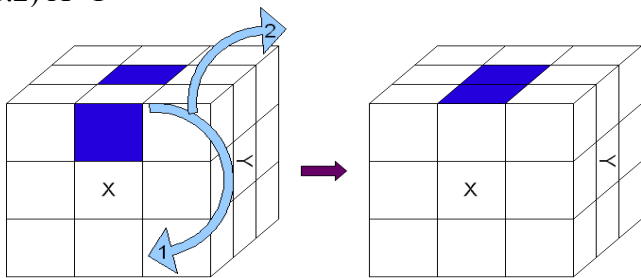

Fig S0- scrambled cube

Stage 1:

We first try to get the cross on the Up face as shown on the left. This is achieve by the following moves:

1.1) X^{-1}, or $(X^{-1})^2$, or X; simply rotate the piece back into place using one of the 3 move variations depend on piece position on edges surounding X. In case below, move used is X^{-1}.

1.2) $X*Y$

We now apply such moves to scrambled cube of fig. S0
The sequence $(F^{-1})*(B)*(R^{-1})*(U^{-1}*F^{-1})$ thus gives us

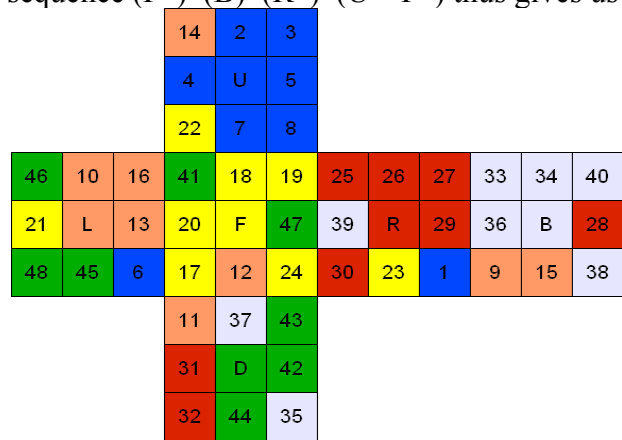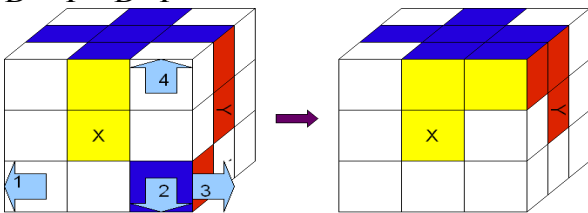|			14	2	3						
			4	U	5						
			22	7	8						
46	10	16	41	18	19	25	26	27	33	34	40
21	L	13	20	F	47	39	R	29	36	B	28
48	45	6	17	12	24	30	23	1	9	15	38
			11	37	43						
			31	D	42						
			32	44	35						

Fig S1- crossed-top scrambled cube

After we have the cross in place, we can put the corners of the Up face back into place using the following moves:

1.3) $D^{-1}*Y^{-1}*D*Y$

1.4) $D*X*D^{-1}*X^{-1}$
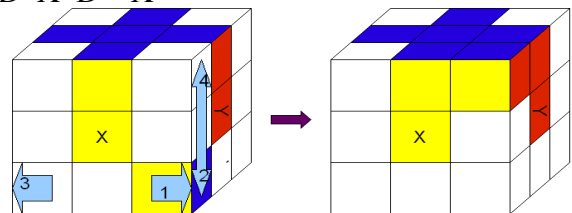
The following face swapping sequence will move the U corner facet to position 1.4 if we place it directly under the corner it should move to.

1.5) $D*X*(D^{-1})^2*X^{-1}$
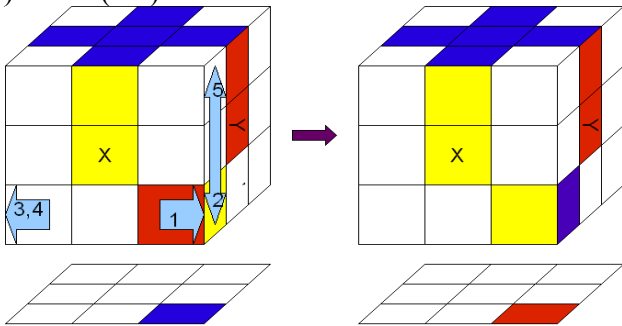
Apply moves 1.3-1.5 to S1, we have:
$(D^{-1}*F^{-1}*D*F)*D^2*(D^{-1}*L^{-1}*D*L)$

			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
20	L	37	12	F	47	39	R	29	36	B	13
14	15	38	48	21	43	24	45	16	22	23	40
			32	28	30						
			44	D	31						
			46	42	41						

Fig S2- Stage 1 complete scrambled cube

We look for the following pattern and applies the appropriate edge moving sequence.

2.1) $D^{-1}*Y^{-1}*D*Y*D*X*D^{-1}*X^{-1}$

2.2) $D*X*D-1*X^{-1}*D^{-1}*Y^{-1}*D*Y$

Applying stage 2 moves to S2, we have:
($D^{-1}*R^{-1}*D*R*D*F*D^{-1}*F^{-1}$)

			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
20	L	37	12	F	21	28	R	29	36	B	13
38	31	22	41	15	43	24	23	40	14	39	32
			16	44	30						
			45	D	42						
			48	47	46						

Fig S3- Stage 2 partial scrambled cube.

At this configuration, we run into a problem because we do not have one of the two two configurations necessary to complete stage 2. Inorder to proceed, we must relocate facets 12+37 and 20+13 into positions of 15+44, 23+42, 39+47, or 31+45 without disturbing the progress we've made so far.

Sequence 2.1 can be applied with X = B, Y = L.
($D^{-1}*L^{-1}*D*L*D*B*D^{-1}*B^{-1}$)

			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	L	44	15	F	21	28	R	29	36	B	37
14	23	48	32	20	43	24	47	41	16	45	40
			38	13	30						
			42	D	39						
			46	31	22						

Fig S5 – Stage 2 partial scrambled cube.
Now with the positioning of facets 13+20, we can apply stage 2 sequence again.

Notice that sequence 2.1 & 2.2 both swaps a middle edge piece with the one across from it, which is precisely what we wanted, so apply 2.2 with Y = F, X = L, we have: ($D*L*D-1*L^{-1}*D^{-1}*F^{-1}*D*F$)

			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
20	L	44	15	F	21	28	R	29	36	B	13
14	39	22	41	23	48	32	45	24	30	37	40
			16	42	38						
			47	D	31						
			46	12	43						

Fig S4 – Stage 2 partial scrambled cube.
Now with the positioning of facets 12+37, we can apply stage 2 sequence again.

Sequence 2.2 can be applied with X = L, Y = F.
($D*L*D-1*L^{-1}*D^{-1}*F^{-1}*D*F$)

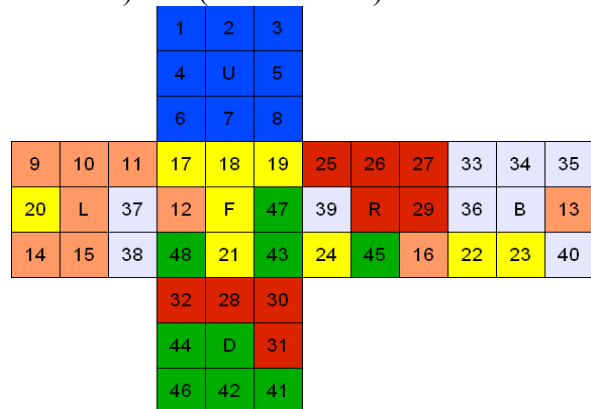|			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	L	13	20	F	21	28	R	29	36	B	37
16	45	48	32	47	46	40	42	24	30	44	41
			38	39	14						
			31	D	23						
			22	15	43						

Fig S6 – Stage 2 complete scrambled cube.

Stage 3:
We again try to achieve the cross as in stage 1. However, the simple sequence of stage 1 would not work here because it will destroy previous results. It should be recognized that the Down face can only fall into one of the following 4 patterns:
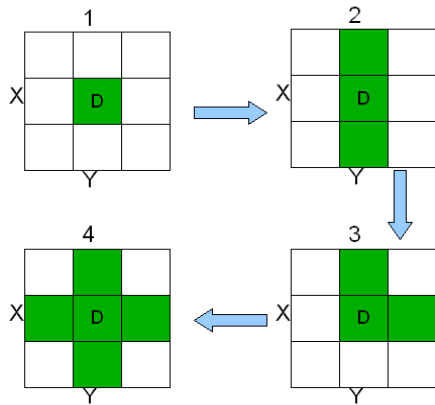
Since our cube in S6 falls into pattern 1, it will take us 3 applications of sequence 3.1 to achieve pattern 4, starting at pattern 0, there is no orientation, so we can pick any side being X. For simplicity, we'll choose X = F, Y = L:
$(F*D*L*D^{-1}*L^{-1}*F^{-1})$
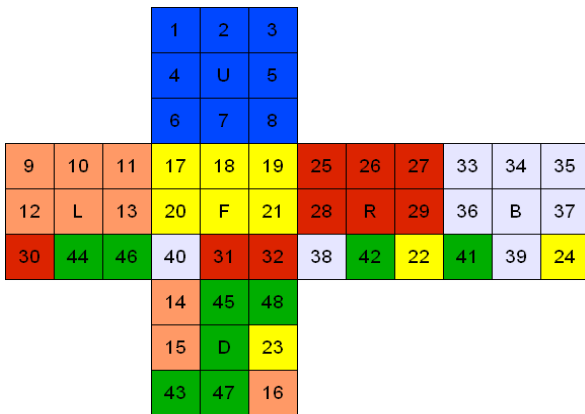
Fig S7 – Stage 3 partial pattern 2

Now that we have pattern 3. Using the orientation of the pattern as reference, we must choose X = R, Y = F
$(R*D*F*D^{-1}*F^{-1}*R^{-1})$

Therefore, once the pattern is recognized, the following sequence can be applied repeatedly to go from one pattern to the next, ultimately resulting in pattern 4.

3.1) $X*D*Y*D^{-1}*Y^{-1}*X^{-1}$
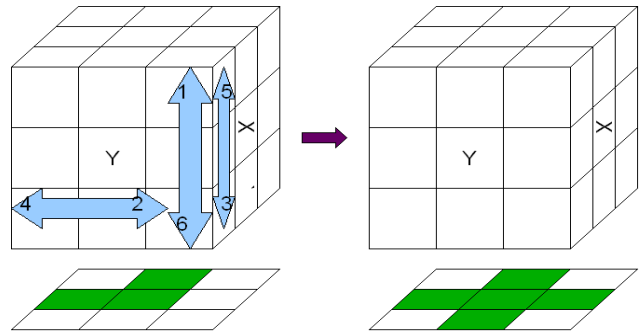
We now have pattern 2. Using the orientation of the pattern as reference, we can choose X = R, Y = F:
$(R*D*F*D^{-1}*F^{-1}*R^{-1})$
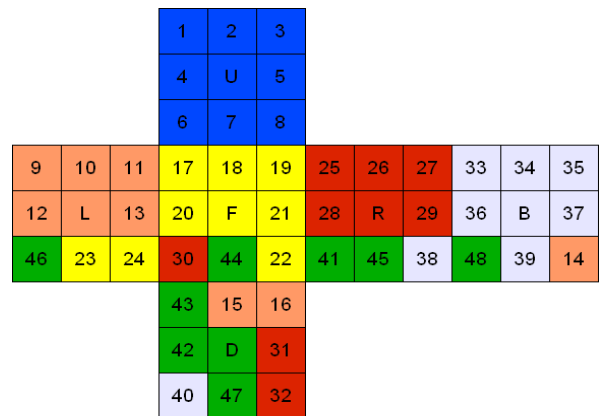
Fig S8 – Stage 3 partial pattern 3

The next stage now involves re-orienting the middle Down facets. From S9, we can see that 42 and 47 are in the correct positions, 45 and 44 are not. Reorienting also requires a special sequence in order not to disturb what we have achieved. In order to use this sequence, we must first rotate B so that at least one corner piece line up with its colour. That side is now the side opposite X.

$3.2)\ (X^{-1}*D^{-1}*X*D^{-1}*X^{-1}*(D^{-1})^2*X*(D^{-1})^2)$

Fig S9 – stage 3 partial pattern 4

Applying 3.2 with X = F on S9, we have
$(F^{-1}*D^{-1}*F*D^{-1}*F^{-1}*(D^{-1})^2*F*(D^{-1})^2)$
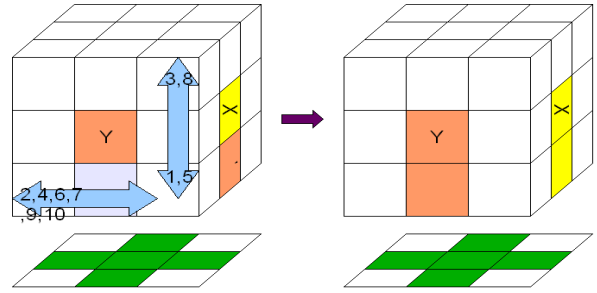

Fig S10 – stage 3 partial

Now that at most 4 corners are out of place, we can employ the following sequence to fix it one by one. At this point, there should be at least 1 corner piece in the correct place. If so, that piece will be on Y on one side and the other side is Z, the side opposite X. If no correct corner piece at this point, pick any.

$3.3)\ (X^{-1}*D*Z*D^{-1}*X*D*Z^{-1}*D^{-1})$
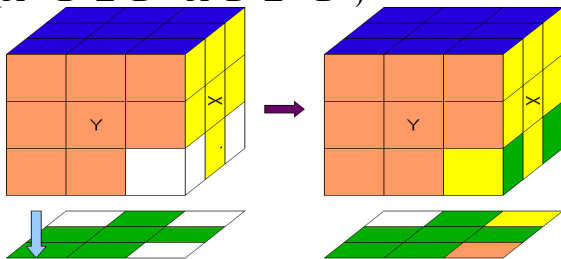
Applying 3.2 again with X = F on S10, we have
$(F^{-1}*D^{-1}*F*D^{-1}*F^{-1}*(D^{-1})^2*F*(D^{-1})^2)$ and
$D\text{-}1*(R^{-1}*D^{-1}*R*D^{-1}*R^{-1}*(D^{-1})^2*R*(D^{-1})^2)*D*D$

Fig S11 – stage 3 partial
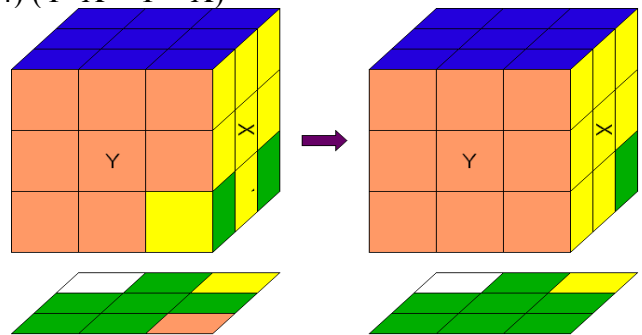
Once the corner pieces are in place and just needed to be rotated, the following sequence can be applied. The corner to fix should be have Y, X, B on 3 sides

$3.4)\ (Y*X^{-1}*Y^{-1}*X)^2$

Reference:

1) History of Rubik's Cube http://www.cs.buffalo.edu/~wli4/rubic/history.html
2) Mathematical solution to 3x3 rubik by Professor W.D. Joyner http://www.cs.buffalo.edu/~wli4/rubic/solve3.txt