Mathematics 307—section 103

First homework solutions

Exercise 1. *Suppose that*

$$f_1 = e_1 + e_2$$
$$f_2 = e_1 + 2e_2$$

(a) If x is a point with

$$x_E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

what is x_F?

```
[
[ 1 1 ]
[ 1 2 ]
]
twoinverse
[
[ 1 ]
[ 1 ]
]
matrixmul
==
(1.a) ==
```

(b) If

$$x_F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

what is x_E?

```
[
[ 1 1 ]
[ 1 2 ]
]
[
[ 1 ]
[ 1 ]
]
matrixmul
==
(1.b) ==
```

Exercise 2. *Suppose that*

$$f_1 = e_1 + e_2 + e_3$$
$$f_2 = e_1 + 2e_2$$
$$f_3 = e_1 - e_3$$

(a) If x is a point with

$$x_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

what is x_F?

```
[
[ 1 1 1 ]
[ 1 2 0 ]
[ 1 0 -1 ]
]
threeinverse
[
[ 1 ]
[ 1 ]
[ 1 ]
]
matrixmul
==
(2.a) ==
```

(b) If

$$x_F = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

what is x_E*?*

```
[
[ 1 1 1 ]
[ 1 2 0 ]
[ 1 0 -1 ]
]
[
[ 1 ]
[ 1 ]
[ 1 ]
]
matrixmul
==
(2.b) ==
```

Exercise 3. *If T is perpendicular projection onto the line $x = y$ what is its matrix? Perpendicular projection onto the line $y = cx$? Perpendicular projection onto the line through the origin and (a, b)?*

A picture shows that the first takes $(1, 0)$ and $(0, 1)$ to $(1/2, 1/2)$, so its matrix is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$M_E = AM_FA^{-1} = \begin{bmatrix} 1 & -c \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -c \\ c & 1 \end{bmatrix}^{-1} = \frac{1}{1+c^2} \begin{bmatrix} 1 & c \\ c & c^2 \end{bmatrix}$$

$$M_E = AM_FA^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

Exercise 4. *If T is perpendicular reflection through the line $x = y$ what is its matrix? Perpendicular reflection through the line $y = cx$? Perpendicular reflection through the line through the origin and (a, b)?*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_E = AM_FA^{-1} = \begin{bmatrix} 1 & -c \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -c \\ c & 1 \end{bmatrix}^{-1}$$

$$M_E = AM_FA^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1}$$

Exercise 5. *Find the matrix of rotation through an angle of 45° around the axis through the line $x = y = z$. Of rotation θ around the same axis.*

```
[[0.804738 -0.310617 0.505879] [0.505879 0.804738 -0.310617] [-0.310617 0.505879 0.804738]]
```

$$M_E = AM_FA^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^{-1}$$

Exercise 6. *Suppose that the f's and e's are as in the first exercise. If a linear transformation has matrix*

$$M_E = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

what is M_F?

```
[
[ 1 1 ]
[ 1 2 ]
]
twoinverse
[
[ 1 2 ]
[ -1 1 ]
]
matrixmul
[
[ 1 1 ]
[ 1 2 ]
]
matrixmul
==
(6) ==

=>

[[6.0 9.0] [-3.0 -4.0]]
(6)
```

Exercise 7. *What is the matrix of perpendicular reflection in the plane $x + 2y + z = 0$?*

```
[
[ 1 1 1 ]
[ 0 -1 2 ]
[ -1 1 1 ]
]
[
[ 1 0 0 ]
[ 0 1 0 ]
[ 0 0 -1 ]
]
matrixmul
[
[ 1 1 1 ]
[ 0 -1 2 ]
[ -1 1 1 ]
]
threeinverse
matrixmul
==
(7) ==
```

Exercise 8. *Classify each of the following matrices A as a (generalized) scaling, rotation, or shear. In each case find a matrix X such that $X^{-1}AX$ has one of the standard forms. In case of a shear, choose the columns of X as orthogonal as possible.*

(a)

$$A = \begin{bmatrix} 8 & 12 \\ -3 & -4 \end{bmatrix}$$

The characteristic polynomial is $\lambda^2 - 4\lambda + 4$, root 2 repeated. The eigenvector equation is

$$6x + 12y = 0$$

so the eigenvector is $(2, -1)$. We have

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

so

$$X = \begin{bmatrix} 12 & 0 \\ -6 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 2+\sqrt{6} & 0 \\ 0 & 2-\sqrt{6} \end{bmatrix}, \quad X = \begin{bmatrix} -1 & -1 \\ 1-\sqrt{6} & 1+\sqrt{6} \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 6 & -3 \\ -1 & 8 & -4 \end{bmatrix}$$

The roots are all 1. The matrix $A - I$ is

$$\begin{bmatrix} 0 & 3 & -2 \\ -1 & 5 & -3 \\ -1 & 8 & -5 \end{bmatrix}$$

which has rank two, so the eigenvectors make up a line.

$$X^{-1}AX = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

(e)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 5 & -2 \\ -3 & 6 & -2 \end{bmatrix}$$

Here the roots are again 1 repeated three times, the matrix $A - I$ is

$$\begin{bmatrix} -1 & 2 & -1 \\ -2 & 4 & -2 \\ -3 & 6 & -3 \end{bmatrix}$$

which as rank one. The eigenvectors form a plane, spanned by $(1,1,1)$ and $(1,2,3)$.

$$X^{-1}AX = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$