Mathematics 266 — Spring 1999 — Part II

Chapter 3. Complex integration

Suppose $f(z)$ to be a complex differentiable function. We want to define for suitable z_0 and z_1 an integral

$$\int_{z_0}^{z_1} f(z)\,dz\;.$$

1. The definition

The basic definition is relatively simple. We choose a parametrized path C from z_0 to z_1, say $t \mapsto z(t)$ with t varying from t_0 to t_1. Then set

$$\int_{z_0}^{z_1} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))z'(t)dt\;.$$

This is the same as the limit of finite sums

$$\sum_1^n f(z_k)(z_k - z_{k-1})$$

as we take finer and finer partitions of the path C into intervals $[z_0, z_1]$, etc. and therefore depends only on the curve, not on the particular parametrization. The reason these two definitions agree is that, given the parametrization, the sum is equal to

$$\sum_1^n f(z(t_k))(z(t_k) - z(t_{k-1})) = \sum_1^n f(z(t_k))\left(\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1})$$

which becomes

$$\int_{t_0}^{t_1} f(z(t))z'(t)dt$$

as the intervals become small.

2. Path independence

In fact, to a large extent this integral depends only on the end points and not even on the curve C.

Suppose that $\alpha(t)$ and $\beta(t)$ are two paths from z_0 to z_1. Suppose that $f(z)$ is defined and complex differentiable in a region including the two paths. Then

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz\;.$$

We can put α and β together, with β going backwards, to make a single closed path, going back to its starting point. So another way of saying this is that if γ is a closed path, then

$$\int_\gamma f(z)\,dz = 0\;.$$

The reasoning explaining this relies on theorems from vector calculus. Recall that if $v(P) = [v_x(P), y_y(P)]$ is a vector field and

$$\gamma\colon t \mapsto P(t) = (x(t), y(t))$$

a parametrized path then the circulation of v around the path is

$$\int_\gamma (v \bullet \mathbf{t})\, ds = \int_{t_0}^{t_1} v(P(t)) \bullet [dx(t)/dt, dy(t)/dt]\, dt = \int_\gamma v_x\, dx + v_y\, dy \ .$$

and by Stokes' Theorem this is also the area integral

$$\int\int_A (\partial v_y/\partial x - \partial v_x/\partial y)\, dx\, dy$$

if A is the inside of γ.

If $f(z) = X + iY$ then the complex integral around a closed path γ is

$$\int_\gamma (X + iY)(dx + idy) = \int_\gamma X\, dx - Y\, dy + i \int_\gamma Y\, dx + X\, dy \ .$$

By Stokes' Theorem this is the same as the area integral

$$\int_A (\partial Y/\partial x + \partial X/\partial y)\, dx\, dy + i \int_A (\partial X/\partial x - \partial Y/\partial y)\, dx\, dy \ .$$

Now since the function $f(z)$ is complex differentiable, the matrix

$$\begin{bmatrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial y \end{bmatrix}$$

represents multiplication by a complex number. Therefore the terms in the area integral vanish.

3. Explicit integration

We can even calculate integrals by using this fact:

Suppose $F(z)$ is a complex differentiable function with $F'(z) = f(z)$, defined throughout a region containing the chosen path from z_0 to z_1. Then the integral along that path is

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0) \ .$$

We shall see later roughly why this is true.

Exercise 3.1. *Find formulas for*

$$\int_0^w e^z\, dz$$
$$\int_0^w \cos z\, dz$$
$$\int_1^w z^n\, dz \quad (n \neq -1)$$

But we have to be very careful. If we take $f(z) = 1/z$ then

$$\int_1^{-1} f(z)\, dz$$

can be evaluated by choosing the curve C to be either the lower or upper half of the unit circle.

Exercise 3.2. *Carry out the two different calculations of*

$$\int_{1}^{-1} \frac{1}{z}\,dz$$

in detail.

Exercise 3.3. *Let* $f(z) = (z^2 - 1)/z$*. Evaluate*

$$\int_{-1}^{1} f(z)\,dz$$

along two different circular arcs.

Exercise 3.4. *Why can't we just write*

$$\int_{1}^{w} \frac{1}{z}\,dz = \log w \text{ ?}$$