Chapter 3. Examples of flows

If $v(x, y) = v(P)$ is a vector field (say in 2D), let $F_t(P)$ be the flow determined by it. Thus for each time t and each point P, $F_t(x, y)$ is the point where (x, y) has moved to in the time interval $[0, t]$. (1) Since at $t = 0$ the point (x, y) has not moved, $F_0(x, y) = (x, y)$. (2) The connection between the flow and the vector field is that the velocity at any moment is equal to the vector in the field at the current location of the point. This means that $(d/dt)F_t(P) = v(F_t(P))$ at all times. These two conditions can be reformulated in this way: Fix $P = (x_0, y_0)$. For any time t let $(x(t), y(t)) = F_t(x_0, y_0)$. Then (1) $(x(0), my(0)) = (x_0, y_0)$. (2) $[x'(t), y'(t)] = v(x(t), y(t))$. In other words, $(x(t), y(t))$ is a solution of a pair of differential equations with initial value (x_0, y_0).

• *Finding the flow for a given vector field is equivalent to solving a system of differential equations.*

Of course you know that solving systems of differential equations is neither easy nor pleasant. In this course we shall be interested only in simple examples.

1. Constant flows

In a constant flow, the velocity is the same everywhere, say $[a, b]$. We are solving the differential equation

$$x' = a$$
$$y' = b$$

and get the flow

$$F_t(x_0, y_0) = (x_0 + at, y_0 + bt) \ .$$

The flow just translates points at a uniform velocity.

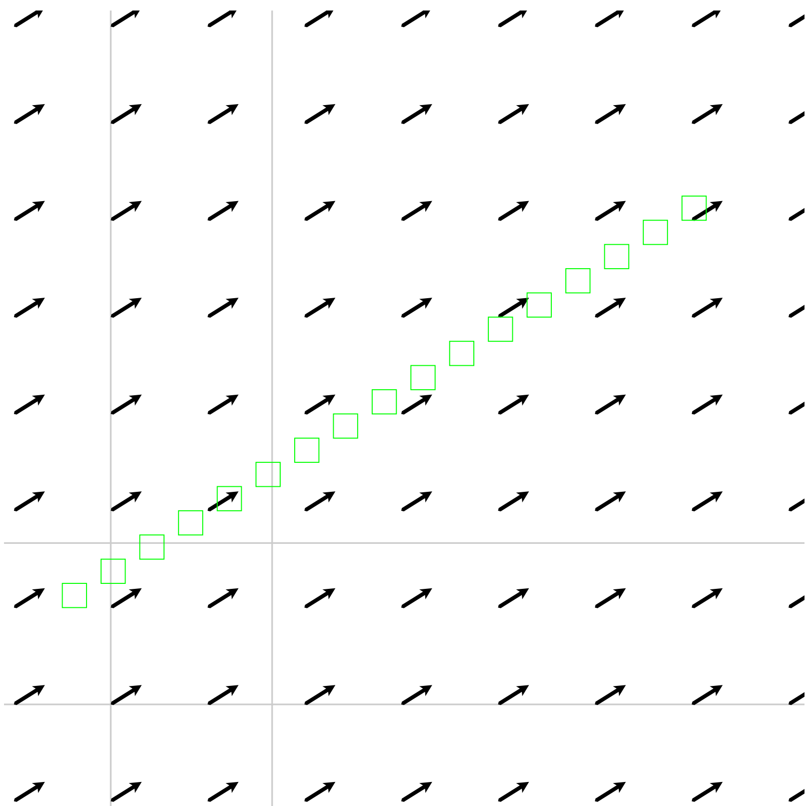

Figure 1

Exercise 1.1. *What are a and b in Figure 1?*

2. Simple rotational flows

Here, the flow rotates every point by a fixed angular velocity, say ω. The system of equations is

$$\begin{aligned} x' &= -\omega y \\ y' &= \quad \omega x \end{aligned}$$

and the flow is

$$F_t(x_0, y_0) = (x_0 \cos \omega t - y_0 \sin \omega t, x_0 \sin \omega t + y_0 \cos \omega t) \ .$$

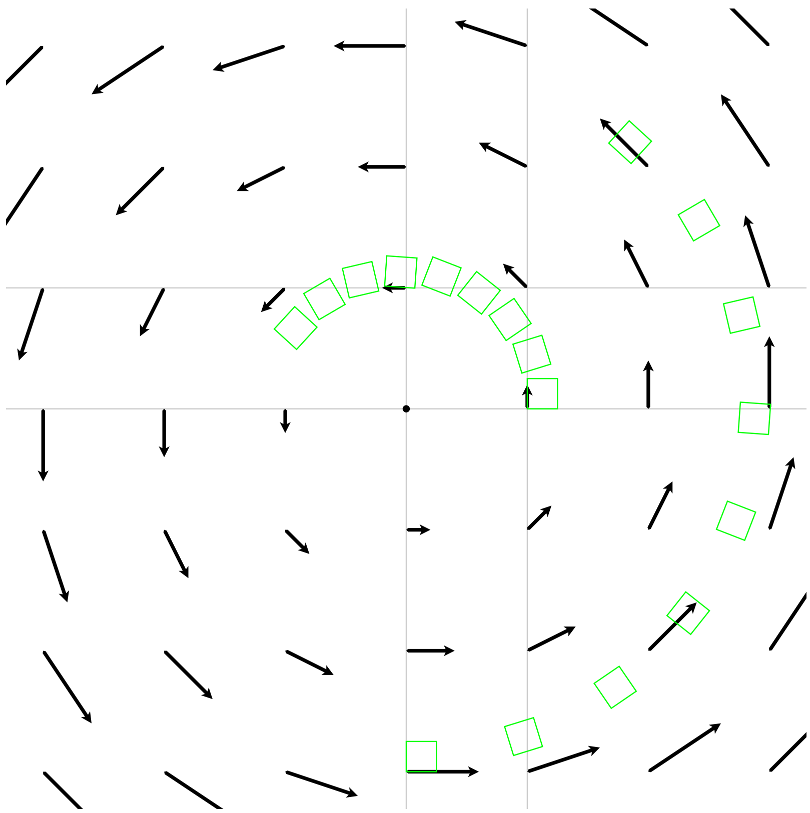

Figure 2

3. A shear flow

A horizontal shear displaces all points horizontally, but the amount of displacement is proportional to y.

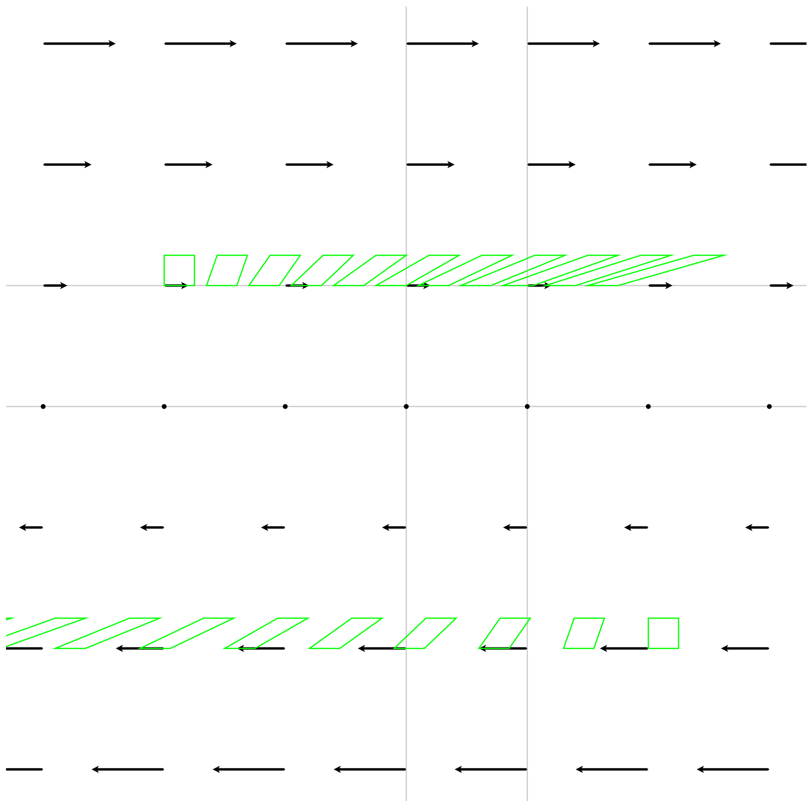

Figure 3

Exercise 3.1. *Write down the differential equations to match Figure 3. What is the flow?*

4. Hyperbolic flows

The velocity vector at (x, y) is

$$[ax, by]$$

where a and b have different signs.

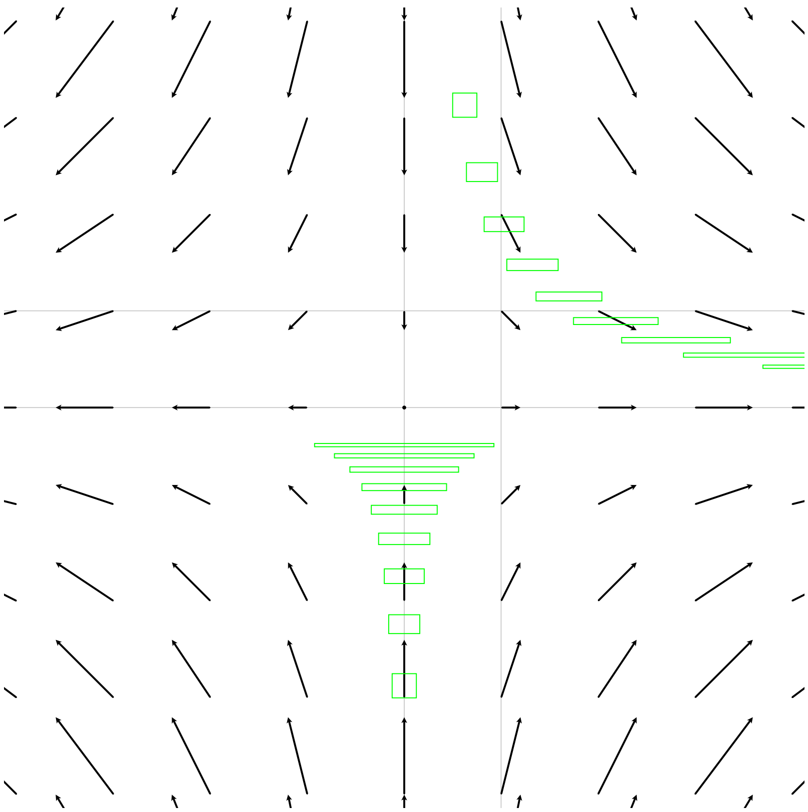

Figure 4

Exercise 4.1. *What is the flow for* $v = [ax, by]$ *(with general* a*,* b*)?*

Exercise 4.2. *What are* a*,* b *in Figure 4?*

5. Parabolic flows

The velocity vector at (x, y) is

$$[ax, by]$$

where a and b have the same sign.

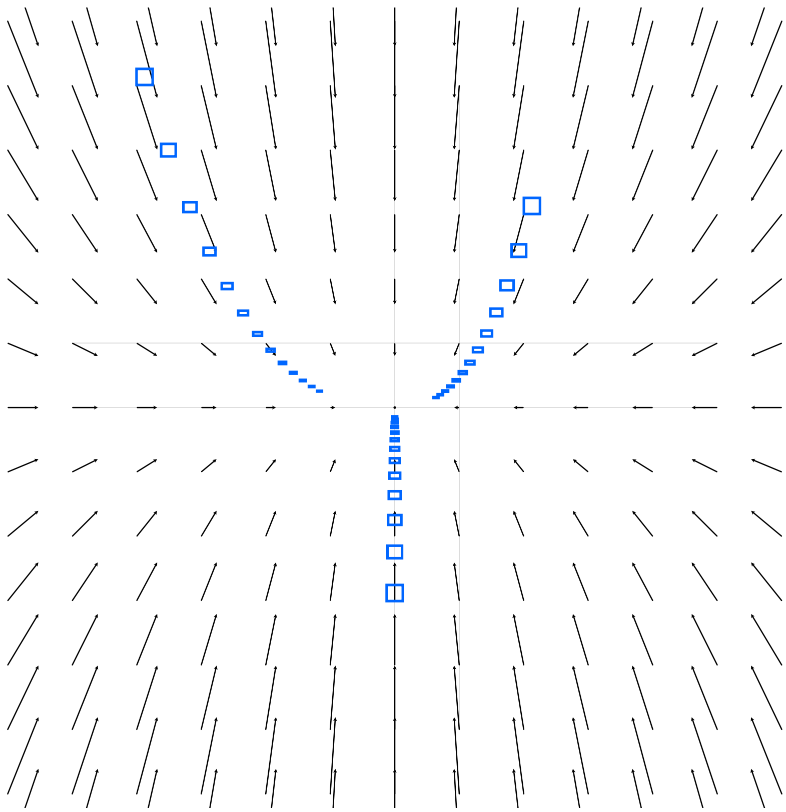

Figure 5

Exercise 5.1. *What are a, b in Figure 5?*

6. An oscillating flow

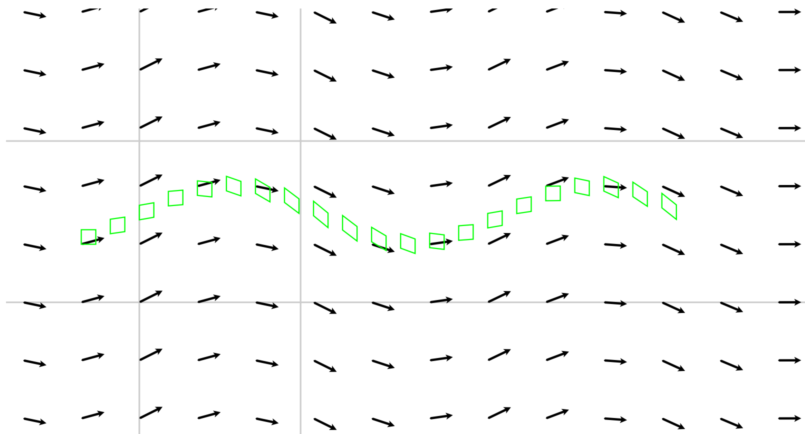

Figure 6

Exercise 6.1. *Write down a plausible guess for the vector field in Figure 6. For the flow.*

7. A non-uniform rotation

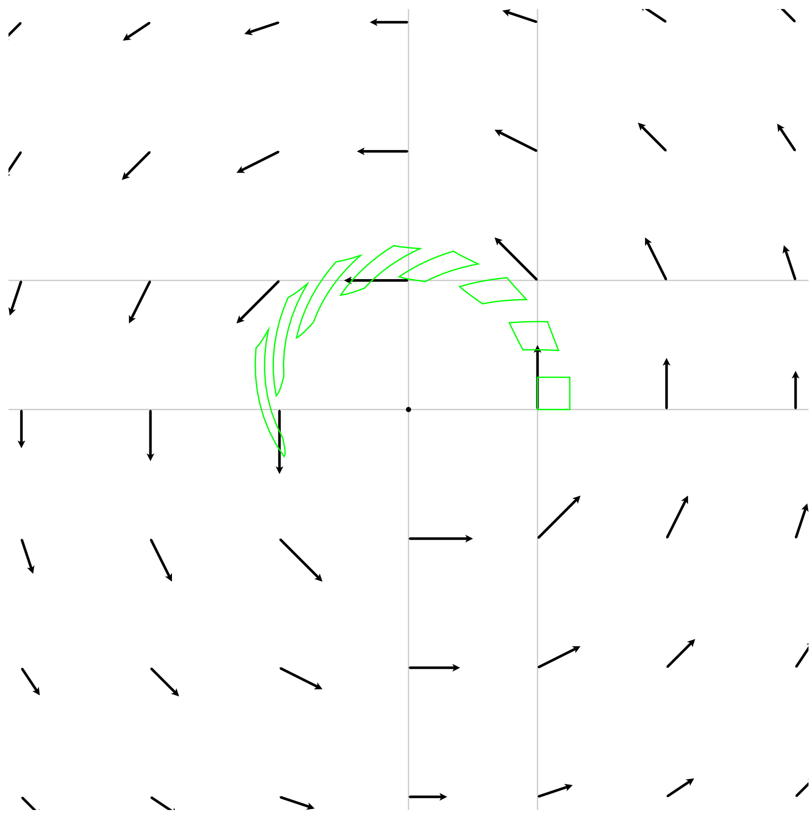

Figure 7

Exercise 7.1. *Write down a plausible guess for the vector field in Figure 7. For the flow.*