Mathematics 256 — Fall 1997 — Section 101

Second homework – due next Wednesday, September 17

1. Solve the following initial value problems for $y = y(x)$:

(i) $y' + 2y = xe^{-2x}$ with $y(1) = 0$.

(ii) $y' + 2xy = 2xe^{-x^2}$ with $y(0) = 0$.

(iii) $y' = 2(1+x)(1+y^2)$ with $y(0) = 0$.

(iv) $y' = 2x/(1+2y)$ with $y(2) = 0$.

2. In a murder investigation a corpse was found by Inspector Clouseau at exactly $8:00$ p.m. Being alert, he measures the temperature of the body and finds it to be $70°$F (Fahrenheit). Two hours later, Inspector Clouseau again measures the temperature of the corpse and finds it to be $60°$ F. If the room temperature is $50°$ F, and we assume that Newton's law of cooling applies, when did the murder occur? (Assume that the temperature of the body at the time of the murder was $98.6°$).

3. In the previous problem, Inspector Clouseau concluded that the time of the murder was 2.6 hours before he took the first temperature reading, or at $5:23$ p. m. However, someone points out that Clouseau's analysis is faulty because the room temperature in which the corpse was found was not constant but instead decreased exponentially according to the law $50e^{-.05t}$, where t is the time (in hours) starting from $8:00$ p.m. Assume that the room temperature obeys this law.

(i) What is the differential equation that Inspector Clouseau must solve now?

(ii) What is the temperature of the body at any time t?

(iii) When was the time of the murder?

4. Consider the differential equation

$$\frac{dy}{dt} = -ky + A\cos(\omega t)\,, \tag{6.1}$$

where $k > 0$ and A are real constants. Replace $\cos(\omega t)$ by $e^{i\omega t}$ in (6.1) to get a problem for $\tilde{y}$

$$\frac{d\tilde{y}}{dt} = -k\tilde{y} + Ae^{i\omega t}\,. \tag{6.2}$$

Look for a solution to (6.2) in the form $\tilde{y}_p = Be^{i\omega t}$ for some B. Then a solution to (6.1) is $y_p = \text{Re}(\tilde{y}_p)$. Here y_p is called the *particular* solution and represents the *steady-state* response for (7.1).

(i) Calculate B and y_p explicitly.

(ii) A solution to (6.1) when $A = 0$ is $y_h = Ce^{-kt}$ for any C. Here y_h is called the *homogeneous* solution and it represents the *transient* response. Thus, the general solution to (6.1), which can satisfy any initial condition is $y = y_h + y_p$. Use the general solution to satisfy $y(0) = 1$ and plot this solution when $\omega = k = A = 1$.

5. A large building with a relaxation time of 1 day has neither internal heating nor cooling but instead responds to the outside air temperature. The outside temperature varies as a sine function, reaching a minimum of $40°$F at $2:00$ a.m. and a maximum of $90°$F at $2:00$ p. m.

(i) Find the initial value problem for the temperature inside the building. (Hint: let t denote time in days with $t = 0$ starting at $8:00$ a.m.)

(ii) Find the steady-state solution of the differential equation found in (i).

(iii) Find the maximum and minimum temperature inside the building over a one day interval.

6. A cake, with a relaxation time of 1 hour, is taken out of an oven at 300°F. It is then allowed to cool in the kitchen where the temperature is 70°F for 30 minutes. At that time the cake is placed into a freezer where the temperature is 20°F.

(i) Find the initial value problem that describes the temperature of the cake at any time t.

(ii) What is the temperature of the cake 30 minutes after it was placed in the freezer?

7. A spherical raindrop evaporates at a rate proportional to its surface area. If the radius of the raindrop is initially 3 mm and after $1/2$ of an hour later it has been reduced to 2 mm, find the time at which the raindrop has completely evaporated.

8. The growth of a three-dimensional cell depends on the flow of nutrients through its surface. Let $W(t)$ be the weight of the cell at time t and W_0 its weight at $t = 0$. Assume that $W^{'}(t)$ is proportional to the area of the cell surface, and that t is measured in hours.

(i) Give an argument to support the proposition that $W^{'} = kW^{2/3}$, where k is a constant.

(ii) If $W_0 = 1$ gram and $W = 8$ grams when $t = 2$ hours, calculate W when $t = 4$ hours.

9. A law that holds over a greater temperature range than Newton's law of cooling is Stephan's law, which states that the temperature T of a material at time t satisfies

$$T^{'} = -k\left(T^4 - T_{\text{env}}^4\right) \qquad (*)$$

where $k > 0$ is constant and T_{env} is the constant environmental temperature. This law holds for bodies heated to a very high temperature that will emit blackbody radiation (i.e. molten iron).

(i) By separating variables and factoring $T^4 - T_{\text{env}}^4 = (T^2 - T_{\text{env}}^2)(T^2 + T_{\text{env}}^2)$, show that the general solution to this equation is

$$\ln\left(\frac{T + T_{\text{env}}}{T - T_{\text{env}}}\right) + 2\tan^{-1}\left(\frac{T}{T_{\text{env}}}\right) = 4T_{\text{env}}^3 kt + c$$

where c is a constant.

(ii) Let $T(0) = T_0 > T_{\text{env}}$. Calculate $\lim_{t\to\infty} T(t)$.

Optional problems

1. Now consider a temperature problem that has a heating or cooling source. It can be written as

$$T^{'} = -k(T - T_{\text{env}}) + H(t)\,,$$

where $H(t)$ denotes either a heating or a cooling source. As an example consider a solar heating problem. A building has a solar heating system that consists of a solar panel and a hot water tank. The tank is well-insulated and has a relaxation time of 50 hours. Under sunlight the energy generated by the solar panel will increase the water temperature in the tank at a rate of 2°F per hour, provided that there is no heat loss from the tank. Suppose that at $9:00$ a. m. the water temperature is 100°F and the room temperature where the tank is stored is a constant 70°F.

(i) Find the initial value problem that describes the temperature of the water in the tank.

(ii) Find the temperature of water in the tank at any time t.

(iii) Find the temperature of the water in the tank after 8 hours of sunlight.

2. John and Mary are having dinner and each orders a cup of coffee. John cools his coffee with some cream. They wait 10 minutes, and then Mary cools her coffee with the same amount of cream. The two then begin to drink. Who drinks the hotter coffee?

3. A ball of mass m falls from rest from a height h towards the ground. We assume that the ball is acted upon by a constant gravitational force and by an opposing frictional force, which is proportional to the square of the velocity. Thus, the velocity $v = v(t)$ (with $v > 0$ if the ball is falling downwards) satisfies

$$m\frac{dv}{dt} = mg - kv^2\,, \qquad v(0) = 0\,,$$

where $k > 0$ is a constant.

(i) Calculate the velocity at any later time t before the ball hits the ground.

(ii) Give a formula (in terms of an integral) for the time at which the ball hits the ground.

4. Torricelli's equation describes the rate at which the level of fluid drops from a leaking tank. More specifically, if a tank has a hole with the area a at its bottom and if $A(y)$ denotes the horizontal cross-sectional area $A(y)$ of the tank at depth y, then the rate at which the liquid drops in the tank is given by Torricelli's equation

$$A(y)\frac{dy}{dt} = -\alpha a\sqrt{2gy}\,, \tag{16.1}$$

where g is the acceleration due to gravity (see Fig. 1) and α is a constant dependent on the type of fluid. Note that the left side of (16.1) represents the change in the volume of fluid in the tank.

Consider a hemispherical tank of radius R with cross-section as shown in Fig. 2. Assume that the tank is initially completely filled with water. A leak is formed when a small circular hole of radius r_0 is punctured at the bottom of the tank.

(i) Show that the differential equation that describes the height y of water in the tank is

$$\frac{dy}{dt} = \frac{-\alpha\pi r_0^2\sqrt{2gy}}{\pi x^2}\,.$$

(ii) Setting $x^2 = R^2 - (y-R)^2 = 2yR - y^2$, and separating variables in the equation above, find an implicit formula for y as a function of t. Use the initial condition $y(0) = R$ (i. e. a full tank of water).

(iii) Show that the time it takes for all the water to drain out is

$$t = \frac{14}{15}\left(\frac{R^{5/2}}{\alpha r_0^2\sqrt{2g}}\right)\,.$$