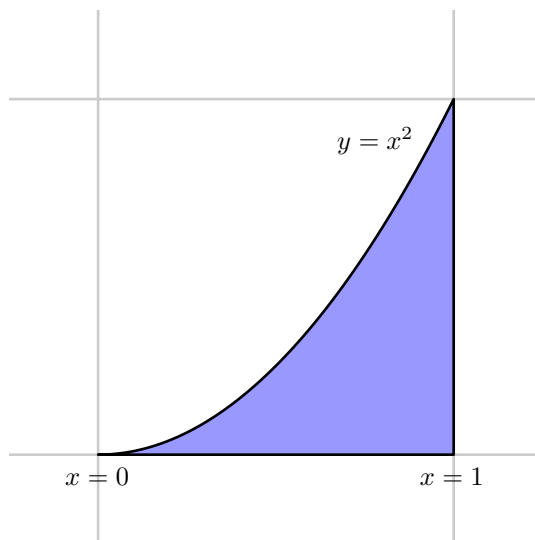
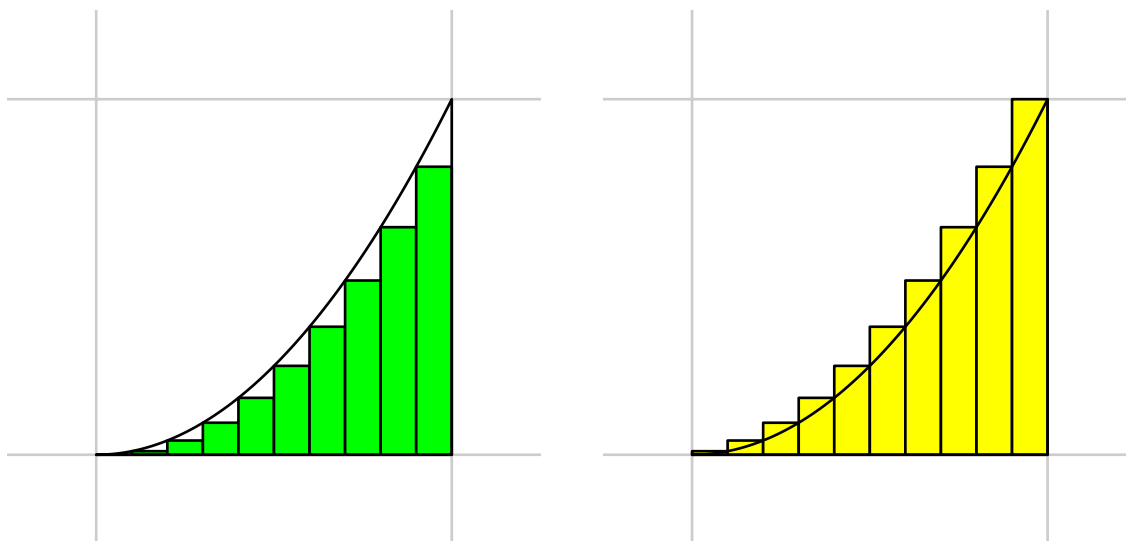


Area under a parabola

The problem we are going to examine in this notes is: *What is the area lying between the graph of $y = x^2$ and the x -axis, between $x = 0$ and $x = a$? We shall begin with the case $a = 1$. So we are asking what the area of of the region shaded below is.*



The technique we shall use is pretty elementary. We first draw inside the parabola a certain number of rectangles. The area inside these rectangles is certainly less than that under the parabola itself. Then we draw the same number of rectangles outside the parabola. The area in the rectangles under the parabola is certainly less than that under the parabola itself, which is also less than that in the rectangles rising above it. Therefore the area under the parabola is 'sandwiched' between the areas of the two sets of rectangles. It turns out that we can calculate these rectangular areas quite explicitly. And finally we shall see what happens as the number of rectangles is taken larger and larger.



The inside rectangles start off with height 0. If we have N rectangles, the value of x increases by $1/N$ in going from one rectangle to the next. The base of the i -th inside rectangle is therefore $1/N$, and its height is therefore

$(i/N)^2$ (the distance up the graph $y = x^2$ above $x = i/N$), and its area is therefore $(1/N)(i/N)^2$. The last rectangle is at $x = (N-1)/N$. The total area of all the inside rectangles is therefore

$$\begin{aligned} & \left(\frac{1}{N}\right) \left(\frac{0}{N}\right)^2 + \left(\frac{1}{N}\right) \left(\frac{1}{N}\right)^2 + \left(\frac{1}{N}\right) \left(\frac{2}{N}\right)^2 + \cdots + \left(\frac{1}{N}\right) \left(\frac{N-2}{N}\right)^2 + \left(\frac{1}{N}\right) \left(\frac{N-1}{N}\right)^2 \\ &= \left(\frac{1}{N}\right) \left(\frac{0^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{1^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{2^2}{N^2}\right) + \cdots + \left(\frac{1}{N}\right) \left(\frac{(N-2)^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{(N-1)^2}{N^2}\right) \\ &= \frac{0^2 + 1^2 + 2^2 + \cdots + (N-1)^2}{N^3} \end{aligned}$$

The outside rectangles start at height $(1/N)^2$ and end at height $1^2 = (N/N)^2$. The total area of all the outside rectangles is therefore

$$\begin{aligned} & \left(\frac{1}{N}\right) \left(\frac{1}{N}\right)^2 + \left(\frac{1}{N}\right) \left(\frac{2}{N}\right)^2 + \cdots + \left(\frac{1}{N}\right) \left(\frac{N-1}{N}\right)^2 + \left(\frac{1}{N}\right) \left(\frac{N}{N}\right)^2 \\ &= \left(\frac{1}{N}\right) \left(\frac{0^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{1^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{2^2}{N^2}\right) + \cdots + \left(\frac{1}{N}\right) \left(\frac{(N-1)^2}{N^2}\right) + \left(\frac{1}{N}\right) \left(\frac{N^2}{N^2}\right) \\ &= \frac{1^2 + 2^2 + \cdots + N^2}{N^3} \end{aligned}$$

The only difference between the two sums is that there is an extra outside rectangle at the end, of height 1. Its area is equal to $(1/N)1^2 = 1/N$. Therefore we have this 'sandwich':

$$\text{area of inner rectangles} < \text{area under parabola} < \text{area under outside rectangles} .$$

or

$$\frac{1^2 + 2^2 + \cdots + (N-1)^2}{N^3} < \text{area under parabola} < \frac{1^2 + 2^2 + \cdots + N^2}{N^3} .$$

Notice that the difference between the two outer terms is

$$\frac{N^2}{N^3} = \frac{1}{N} .$$

The point now is that we know what the sums of the squares are, from a previous section. We have

$$1^2 + 2^2 + \cdots + N^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} .$$

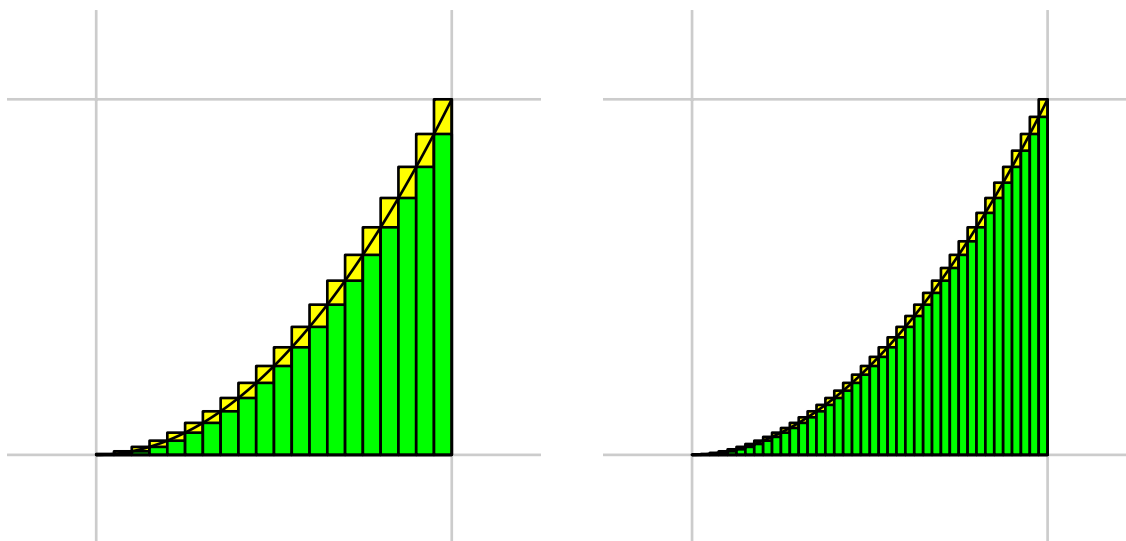
So we have

$$\frac{N^3}{3N^3} - \frac{N^2}{2N^3} + \frac{N}{6N^3} < \text{area under parabola} < \frac{N^3}{3N^3} + \frac{N^2}{2N^3} + \frac{N}{6N^3}$$

or

$$\frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2} < \text{area under parabola} < \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2} .$$

And this holds for any N , no matter how large.



Most of the terms in the outer expressions—all but the first—get very small if N gets very large. Therefore the only way this can be true for all N is if the area under the parabola is exactly $1/3$.

Exercise 1. Find the area under the parabola between $x = a$ and $x = b$ by a similar calculation (with $a < b$).

Exercise 2. Find the area under the graph $y = x^3$ between $x = 0$ and $x = 1$ by a similar calculation.

Exercise 3. Find the area under the graph $y = x^3$ between $x = a$ and $x = b$ by a similar calculation (with $a < b$).