Mathematics 103 — section 203 — Spring 2001
Ninth (and last) homework — due Friday, March 30

Exercise 1. *A biochemical reaction $S \to P$ is catalysed by an enzyme E. The speed of the reaction depends on the concentration of the substrate S. It is found that the substrate concentration x changes at the rate*

$$\frac{dx}{dt} = rx - \frac{Kx}{k+x}$$

where r, K, k are positive constants. (The second term is often called Michaelis-Menten kinetics in chemistry.) How many steady state concentration values are there? What are these values? What happens to the solutions $x(t)$ for very large and for very small values of r? Do sketches of the slope fields and graphs of solutions, one each for r small and r large.

Exercise 2. *Find the second moment of $\mathrm{Erf}(x)$; of $\mathrm{Erf}(x/\sigma)$. This will require integration by parts. You will need to use the fact that the total area under $y = \mathrm{Erf}(x)$ is 1.*

Exercise 3. *Use partial fractions to find*

$$\int \frac{x}{x^2 - 1}\, dx$$
$$\int \frac{1+x}{x^2 + 5x + 6}\, dx$$
$$\int \frac{1 + x + x^3}{x^2 + 5x + 6}\, dx$$

Exercise 4. *Find the general solution to the differential equation*

$$y' = -y^2 + 3y - 2$$

and then the solution with $y(0) = 0$.

Exercise 5. *Tell which of the following integrals is finite, and if so evaluate it.*

$$\int_0^\infty \frac{1}{1+x}\, dx$$
$$\int_0^\infty \frac{1}{1+x^2}\, dx$$
$$\int_0^\infty \frac{1}{1+2x+x^2}\, dx$$
$$\int_0^\infty \frac{1}{1+3x+2x^2}\, dx$$
$$\int_0^\infty e^{-x}\, dx$$
$$\int_0^\infty xe^{-x}\, dx$$
$$\int_0^\infty x^2 e^{-x}\, dx$$

(You will need integration by parts for the last few.)

Exercise 6. *The statistics of radioactivity demonstrate that atoms of a fixed type do not change state with time. That is to say, a radium atom now is virtually identical with a radium atom $1,000$ years ago.*

This means that for any given radium atom the probability that it will decay in the next ten minutes is not different from what it was 1,000 years ago. This is hard to understand, but let's do some calculations to that effect.

Suppose that the probability that for a radioactive atom at time $t = 0$ of a certain type the probability that it will not decay before time t (measured in years) is $e^{-t/\tau}$. (a) What is the probability that it will decay in the first year? (b) What is the probability that it will last at least 2 years? (c) What is the probability that it will last $t + \Delta t$ years? (d) What is the probability that it will decay in between t and $t + \Delta$ years? (e) Among those which last at least t years, what fraction of them will decay in the next period of Δt years? (f) What happens to this as t goes to ∞? (g) What is the half-life of this type of atom? (h) The half-life of radium is 1620 years. Suppose that you are watching a given radium atom. What is the probability that it will decay in the next 100 years? Suppose that 50 years from now you discover that it has not yet decayed. What is the probability that it decays in the next 50 years after that? The next 100?

Exercise 7. We have seen that if a coin is not too unbalanced, then the probability of getting heads in a large number of tosses can still be approximated by areas under the graph of a bell curve. This is no longer true if it is very unbalanced. Instead, if p is small and n is large, the probability of getting k heads in n tosses can be approximated by the quantity $(\mu^k/k!)e^{-\mu}$ where $\mu = np$. Try this out for $n = 20$, $p = 0.1$. Plot bar graphs of the exact probability of getting k heads in 20 tosses, for $k = 0, \ldots 20$; then plot bar graphs of $(\mu^k/k!)e^{-\mu}$.

Exercise 8. Find all the equilibrium solutions of

$$y' = y(1-y) - \frac{cy}{y+d} \ .$$

when $0 < d < 1$ and $c = (d+1)^2/4$. Which are stable? Unstable? Fix $d = 1/2$ and sketch the slope field in the range $0 \le y \le 2$.