The A-polynomials of families of symmetric knots

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Abstract: The A-polynomial $A(x, y)$ (*not* the Alexander polynomial) of a knot complement is an invariant that is notoriously difficult to compute. For example, the A-polynomials of one of the knots 8_{17} with 8 crossings is still unknown. We will describe a method for computing the A-polynomials of knots in a family of branched cyclic coverings of a 2-link X branched over one of the components of the link. The n-fold cover X_n has symmetry group containing the cyclic group Z_n. The main result is a formula for the A-polynomial of X_n which gives $A(x^n, y)$ as a product of $n\phi(2n)/2$ (if n is odd) or $n\phi(2n)/4$ (if n is even) polynomials with coefficients in $\mathbb{Q}(2\cos(\pi/n))$. These polynomials are all obtained from a single polynomial $G(x, y, w)$ called the G-polynomial of the link which is computed from a representation of the fundamental group of the link or from a triangulation of the link complement. A familiar example is the sequence of Turk's head knots with 3 strands of which the knot 8_{18} is the first interesting example. Using this method we compute the A-polynomial of one of the dodecahedral knots by regarding it as a 5-fold branched cyclic covering of a certain 2-bridge link. The resulting polynomial is of degree 32×160 in (x, y) and the largest coefficient is a 32 digit integer. It is unlikely that this polynomial could be computed by any of the earlier methods. We also consider some examples of the analogous process for links with 3 components. In this case one obtains two-parameter families of highly symmetric knots $X_{m,n}$ for which $A(x^{mn}, y)$ is expressed as a product of polynomials with coefficients in $\mathbb{Q}(2\cos(\pi/m), 2\cos(\pi/n))$ all obtained from a four variable polynomial $H(x, y, v, w)$.

The Apolynomials of families of Symmetric knots

K knot in $S^3 \longrightarrow A(x,y) \in \mathbb{Z}[x,y]$

A-polynomial (not Alexander)

Two almost equivalent definitions:

(1) In terms of representations

$\pi_1(K) \longrightarrow SL(2,\mathbb{C})$ (CCGLS '94)

(2) In terms of triangulations of $X = S^3 \setminus K$ into ideal tetrahedra in $\mathbb{H}^3$, hyperbolic 3-space

related to reps

$\pi_1(K) \to PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$

Computing $A(x,y)$ is usually difficult requiring elimination of variables in large systems of polynomial eqns
- resultants
- Gröbner bases

For (1), one starts with a presentation of π_1 in terms of g generators

- in practice this works only if

$g = 2$ (and relns not too long)

For (2), one starts with a triangulation of $S^3 \setminus K$ into t ideal tetrahedra and this works in practice only if

$t \leq 8$ (or so, depending on complexity of eqns)

We recently developed a new method which uses Puiseux expansions & depends on some other quantities being small;

(i) d = degree of the Shape field (e.g. $d \leq 12$)

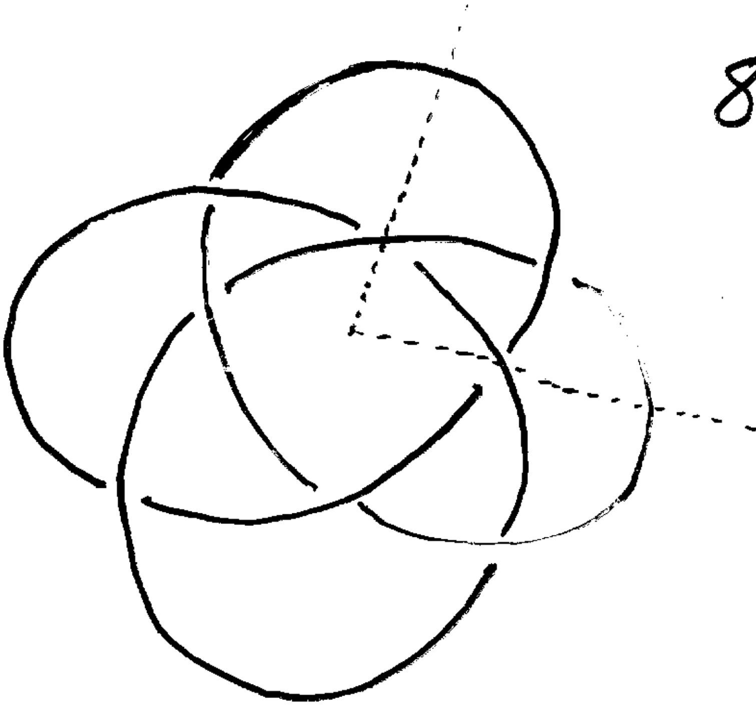
(ii) degree (A, x) and degree (A, y)

e.g. $A(x,y)$ 16×80 or so.

non 2-bridge 8-crossing knots

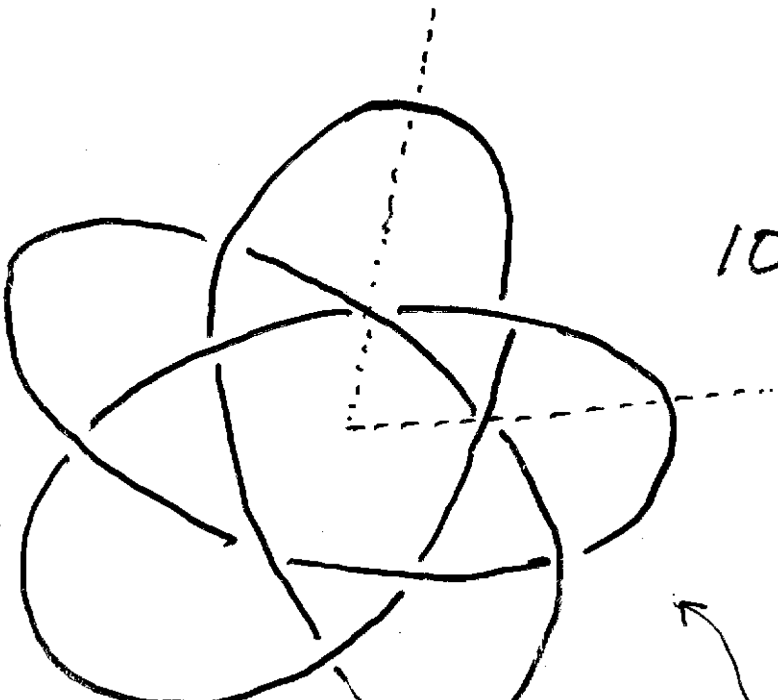
Rolfsen	Conway	#tet	#gen	deg	Sym
8_5	3,3,2	8	2	5	D_2
8_{10}	3,21,2	9	2	11	Z_2
8_{15}	21,21,2	11	2	7	D_2
8_{16}	.2.20	11	3	5	Z_2
8_{17}	.2.2	12	3	18	Z_2
8_{18}	8^*	13	3	4	D_8
8_{19}	3,3,2-	non-hyperbolic			
8_{20}	3,21,2-	5	2	5	Z_2
8_{21}	21,21,2-	7	2	4	D_2

Some Turk's Head knots



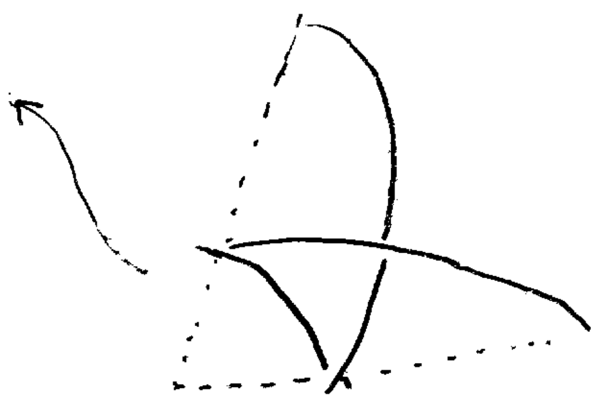
$8^* = 8_{18}$

Sym: D_8
amphicheiral



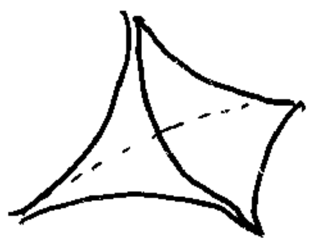
$10^* = 10_{123}$

Sym: D_{10}
amphicheiral



The gluing equations

If $X = S^3 \setminus K = \bigcup_{k=1}^{t} \Delta_k$, each

$\Delta_k =$  an ideal tetrahedron in $\mathbb{H}^3$ = hyperbolic 3-space. $\cong \mathbb{C} \times (0, \infty)$

Up to isometry,

$\Delta_k \cong \Delta(z)$ with vertices at $0, 1, z, \infty$, $z \in \mathbb{C}$

$\mathrm{Vol}(\Delta_k) = D(z)$, the Bloch-Wigner dilog.

($\pi\, m(A) = \sum D(\alpha_j)$, certain $\alpha_j \in \overline{\mathbb{Q}}$)

The combinatorics of the triangulation lead to the gluing eqns

$$(G) \quad f_j = \prod_{j=1}^{t} z_j^{a_{ij}} (1 - z_j)^{b_{ij}} = 1, \quad i = 1, \dots, t+2$$

where $f_j = 1$ $(j \le t)$ are "edge" eqns & $f_{t+1} = 1$, $f_{t+2} = 1$ are the "longitude" & "meridian" eqns.

We "deform" the last two eqns to

$$f_{t+1} = x^2, \; f_{t+2} = y$$

and then eliminate $z_1, \dots, z_t$ from the resulting system to obtain $A(x,y) \in \mathbb{Z}[x,y]$

A-polynomials and volume

One of the most interesting aspects of the A poly is that if (x,y) satisfy $A(x,y)=0$ then the form

$$\log|x|\, d\arg y - \log|y|\, d\arg x = dV \text{ is \underline{exact}}$$

(Hodgson, Dunfield, Cooper-Culler-Gillet-Long-Shalen).

From this, one can deduce that

$$\pi m(A) = \pi \int_0^{2\pi}\int_0^{2\pi} \log|A(e^{is}, e^{it})|\, ds\, t$$

$$= \sum_{\alpha} D(\alpha) \qquad (*)$$

for a finite set of algebraic α.

Here $D(z)$ = volume of ideal tetrahedron with vertices at $0, 1, z, \infty$.

This suggests $\pi m(A) =$ "volume"

which sometimes is $\pi m(A) = vol(X)$

or more usually $\pi m(A) = |Ball(X)|_{\ell^1}$

But in general $(*)$ has even more terms than these suggest.

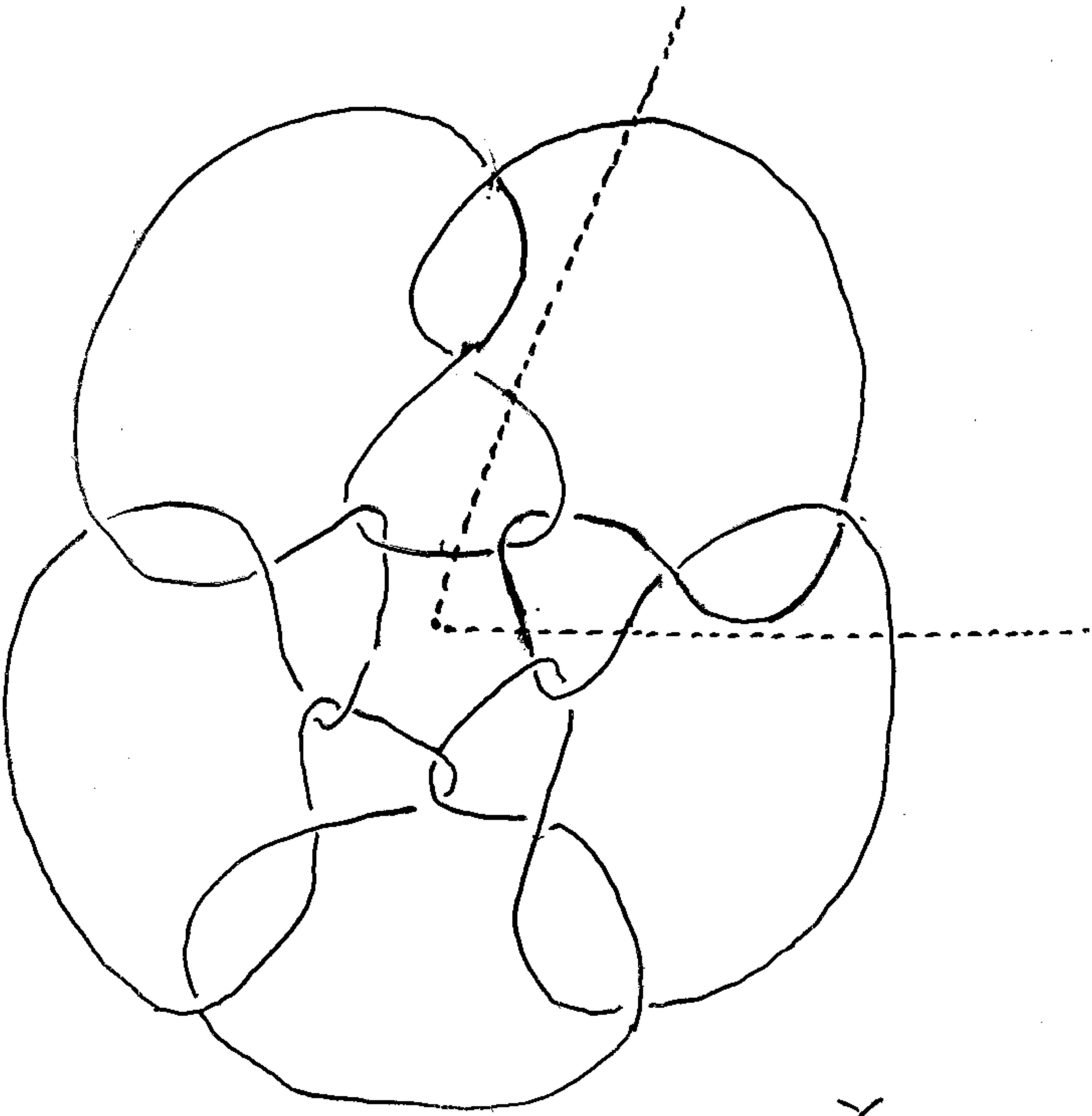
The computation of $A(x,y)$ for $8_{18} = 8^*$ resisted attempts by methods (1) & (2) finally yielding to the new method (3) in July 2003. The related 10^*, 14^*, 16^* and 20^* also yielded to variants of this method.

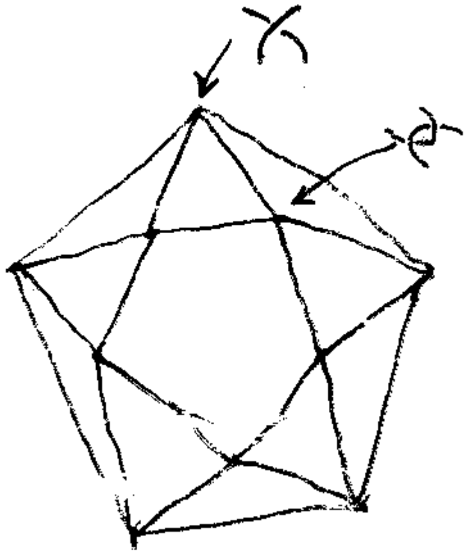
However, a knot like dodec, interesting since it is one of 3 known knots with Cusp field $\neq$ Shape field did not yield to this method, mainly because degree$(A,x) \geq 24$ is too large.

None of these methods exploit the obvious <u>symmetry</u> of the knots.

We'll see that the way to exploit the symmetry is to realize the knots as

cyclic branched covers





$dodec_1$

Sym: D_{10}

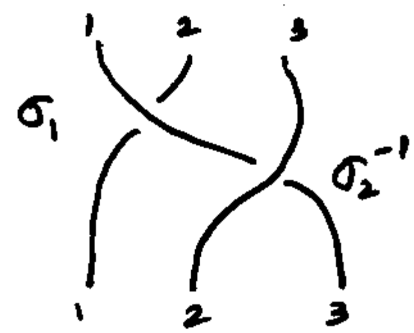
amphicheiral

$g = 4$, $t = 46$, $d = 4$

$\text{degree}(A, x) \geq 24$

It is clear from the picture of 10^* or dodec, that the knot is formed from 5 copies of a "template" rotated through $2\pi/5$.

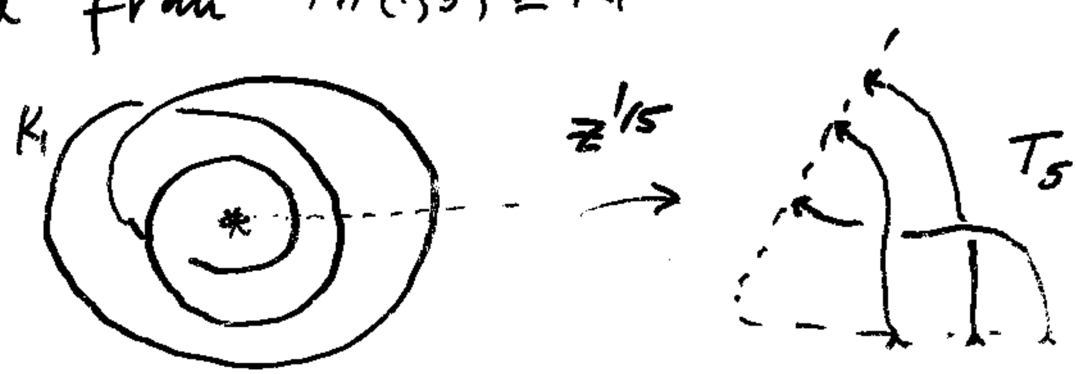
Topologically, the template is just the braid $\sigma_1\sigma_2^{-1}$



and $10^* = [(\sigma_1\sigma_2^{-1})^5]$.

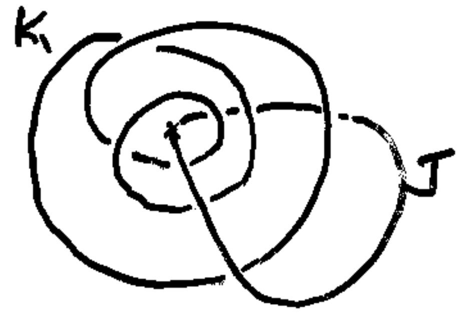
In general $Th(n,3) := [(\sigma_1\sigma_2^{-1})^n]$, a knot if $3 \nmid n$ and a 3-link if $3 \mid n$.

We can think of the template as being obtained from $Th(1,3) = K_1$



followed by pasting together 5 rotated copies of T_5 to get $K_n = (2n)^*$

If one introduces a loop J



← link L

$X = S^3 \setminus L$

then $T_5 = X(5,0)$ ← Dehn surgery along J

is a cone-manifold, hyperbolic except for a cone angle of $2\pi/5$ along J.

And $K_5 = X(5,0)^5$ a cyclic 5-cover of T_5 around J

is a manifold

The singularity along J disappears (along with J itself!)

There is a generalization of the A-poly to links like L that we call the G-poly

$$G(x,y,w) \in \mathbb{Z}[x,y,w]$$

This may be much easier to compute than A since L is basically simpler than any K_n

<u>Theorem:</u> The A poly of $X_n = X(n,0)^n$ is obtained as follows:

$$F_0(x,y) = \prod_{\substack{(k,2n)=1 \\ 1<k<n}} G(x,y,\underbrace{\zeta_{2n}^k + \frac{1}{\zeta_{2n}^k}}_{\omega_{2n}^{(k)}}) \in \mathbb{Z}[x,y]$$

$$B_n(x,y) = \prod_{j=0}^{n-1} F_0(\zeta_n^j x, y) \in \mathbb{Z}[x^n,y]$$

$$B(x,y) = \begin{cases} B_n(x^{1/n}, y), & n \text{ odd} \\ \sqrt{B_n(x^{1/n}, y)}, & n \text{ even} \end{cases}$$

Then $$A(x,y) = B(-x,y).$$

This expresses $A(-x^n, y)$ as a product of $\frac{1}{2}\varphi(2n)\cdot n$ polys $G(\zeta_n^j x, y, \omega_{2n}^{(k)})$

(or $\frac{1}{4}\varphi(2n)\cdot n$ if n even) e.g. $A(x,y)$ for 20^* is of degree 8×80

$$A(-x^{10}, y) = F_1 F_2 F_3 F_4 F_5 \quad \text{each } 16 \times 16 \text{ in } \mathbb{Z}[x,y]$$

e.g. $F_1 = G_1 G_2 G_3 G_4$ in $\mathbb{Q}(\zeta_{2n})[x,y]$.
all 4×4

For the Turk's Head knot.

$$G = \begin{bmatrix} & & 1 & & \\ & w & -1-w^2 & w & \\ 1 & -2w & -2+2w^2 & -2w & 1 \\ & w & -1-w^2 & w & \\ & & 1 & & \end{bmatrix}$$

Relative to Mahler's measure

$$m(A) = 5\, m(F_1) = 5\,(m(G_1) + m(G_2) + m(G_3) + m(G_4))$$

and in fact $\quad Vol(20^*) = 10\pi\, m(G_1)$

Adopting the position that

$$A_1(x,y) = \prod_{j=0}^{9} G_1(x\zeta_n^j, y)\Big|_{x^{10} \to -x}$$

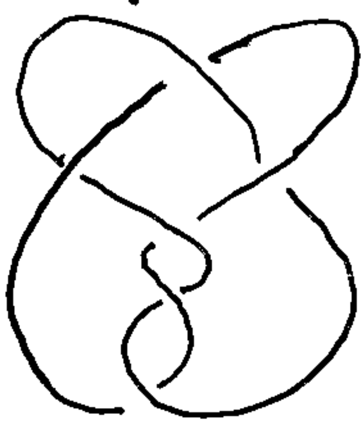
is the "true" Apoly we thus have

$$Vol(20^*) = \pi\, m(A_1)$$

& indeed this is true for all $(2n)^*$ (even if $3 | n$ as it happens)

Computation of G for $6_2^2 = [33]$

The "master link" for the $Tk(n,3)$ is in fact the 2-bridge link $[33] = 6_2^2$ whose complement can be triangulated with 4 ideal tetrahedra:



Gluing eqns

$$f_1 = \frac{z_1 z_2 (1-z_3)(1-z_4)}{(1-z_2)\; z_4} - 1 = 0$$

$$f_2 = \frac{(1-z_1) z_3}{(1-z_2) z_4} - 1 = 0$$

$$f_3 = \frac{(1-z_2) z_4}{(1-z_1) z_3} - 1 = 0$$

$$f_4 = \frac{(1-z_2) z_4}{z_1 z_2 (1-z_3)(1-z_4)} - 1 = 0$$

(f_2, f_3: equivalent; f_1–f_4: edge eqns)

$$f_5 = \mu_1 - 1 = \frac{1-z_2}{1-z_1} - 1 = 0$$

$$f_6 = \mu_2 - 1 = \frac{1}{(1-z_2) z_4} - 1 = 0$$

$$f_7 = \lambda_1 - 1 = \frac{z_1 (1-z_2)(1-z_3) z_4}{(1-z_1) z_2 z_3 (1-z_4)} - 1 = 0$$

$$f_8 = \lambda_2 - 1 = \frac{z_1^2 z_2^2 z_3}{(1-z_2)^2 z_4} - 1 = 0$$

(f_5–f_8: cusp eqns)

We want to deform $\mu_2 = 1$, $\lambda_2 = 1$

to $\mu_2 = y$, $\lambda_2 = x^2$

And since we want to do $(n,0)$ surgery on the 1st component we want

$$\mu_1^n = 1 \quad \& \quad \lambda_1 = \text{whatever}$$

So let $\mu_1 = z^2$ where eventually $z^{2n} = 1$

i.e. $z = \zeta_{2n}$.

Eliminate $z_1, \ldots, z_4$ and we get a polynomial $F(x,y,z)$ of degree $4 \times 4 \times 4$.

F is reciprocal in z i.e. $z^4 F(x,y,z^{-1}) = F(x,y,z)$

so we define $z + z^{-1} = w$ and get

$$G(x,y,w) \quad 4 \times 4 \times 2$$

$$G = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & w & -1-w^2 & w & \cdot \\ 1 & -2w & -2+2w^2 & -2w & 1 \\ \cdot & w & -1-w^2 & w & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

8* Apoly

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 54 & 0 & 0 \\ 0 & 0 & -112 & 0 & 0 \\ 0 & -2 & 109 & -2 & 0 \\ 0 & 12 & -64 & 12 & 0 \\ 0 & -14 & 74 & -14 & 0 \\ 0 & -28 & -100 & -28 & 0 \\ 1 & 68 & 106 & 68 & 1 \\ 0 & -28 & -100 & -28 & 0 \\ 0 & -14 & 74 & -14 & 0 \\ 0 & 12 & -64 & 12 & 0 \\ 0 & -2 & 109 & -2 & 0 \\ 0 & 0 & -112 & 0 & 0 \\ 0 & 0 & 54 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

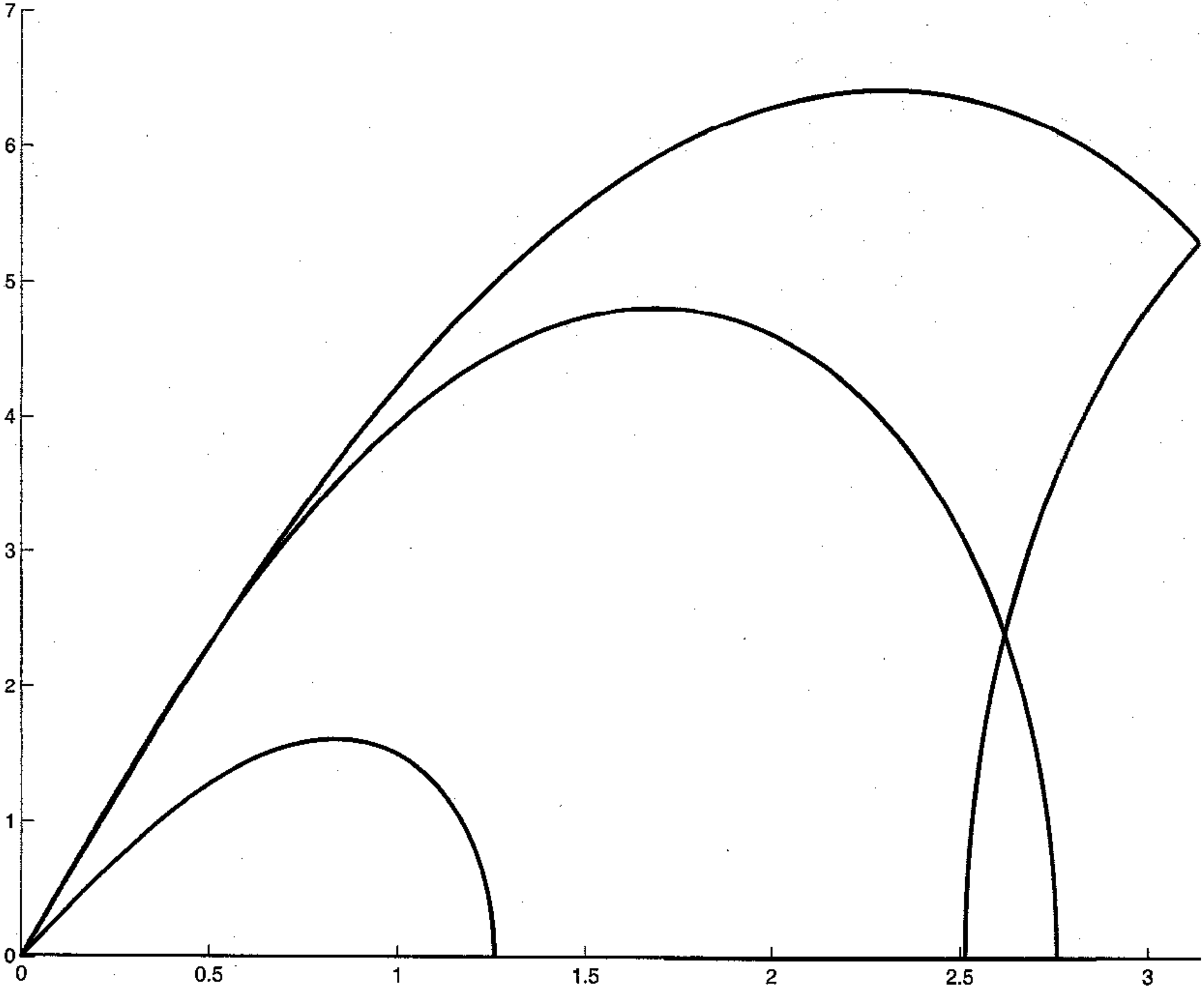
10* A-polynomial

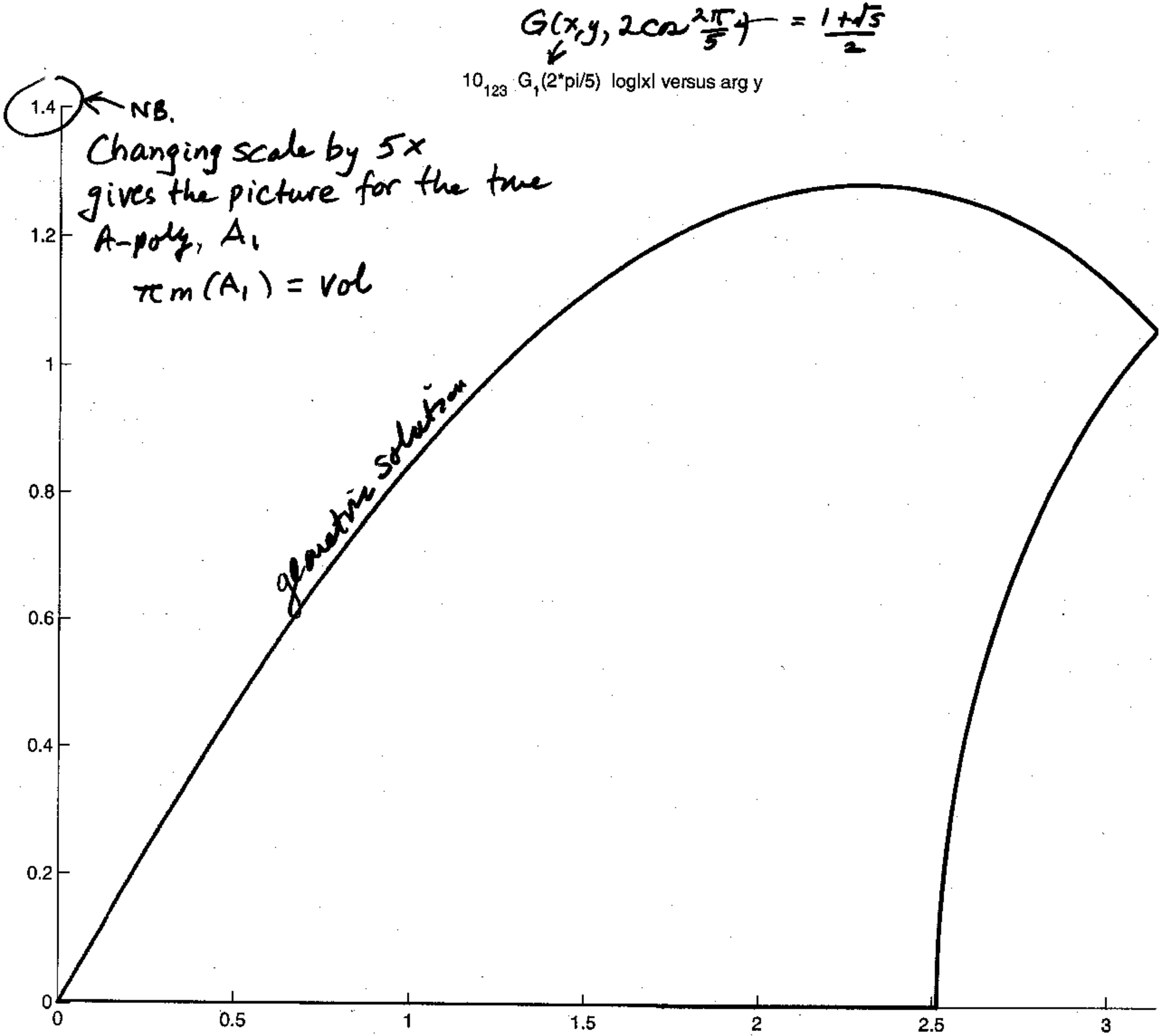
0	0	0	0	1	0	0	0	0
0	0	0	0	-25	0	0	0	0
0	0	0	0	270	0	0	0	0
0	0	0	0	-1640	0	0	0	0
0	0	0	0	6075	0	0	0	0
0	0	0	4	-13710	4	0	0	0
0	0	0	-75	16850	-75	0	0	0
0	0	0	585	-5215	585	0	0	0
0	0	0	-2400	-11290	-2400	0	0	0
0	0	0	5125	7275	5125	0	0	0
0	0	6	-3353	9720	-3353	6	0	0
0	0	-75	-8250	-7400	-8250	-75	0	0
0	0	360	17470	-4740	17470	360	0	0
0	0	-755	-4900	2515	-4900	-755	0	0
0	0	275	-11425	-725	-11425	275	0	0
0	4	1099	-578	6848	-578	1099	4	0
0	-25	1475	17200	-4950	17200	1475	-25	0
0	45	-10910	-7565	-9320	-7565	-10910	45	0
0	5	14080	-5740	18870	-5740	14080	5	0
0	-150	400	7600	-1250	7600	400	-150	0
1	242	-11914	-7396	-16312	-7396	-11914	242	1
0	-150	400	7600	-1250	7600	400	-150	0
0	5	14080	-5740	18870	-5740	14080	5	0
0	45	-10910	-7565	-9320	-7565	-10910	45	0
0	-25	1475	17200	-4950	17200	1475	-25	0
0	4	1099	-578	6848	-578	1099	4	0
0	0	275	-11425	-725	-11425	275	0	0
0	0	-755	-4900	2515	-4900	-755	0	0
0	0	360	17470	-4740	17470	360	0	0
0	0	-75	-8250	-7400	-8250	-75	0	0
0	0	6	-3353	9720	-3353	6	0	0
0	0	0	5125	7275	5125	0	0	0
0	0	0	-2400	-11290	-2400	0	0	0
0	0	0	585	-5215	585	0	0	0
0	0	0	-75	16850	-75	0	0	0
0	0	0	4	-13710	4	0	0	0
0	0	0	0	6075	0	0	0	0
0	0	0	0	-1640	0	0	0	0
0	0	0	0	270	0	0	0	0

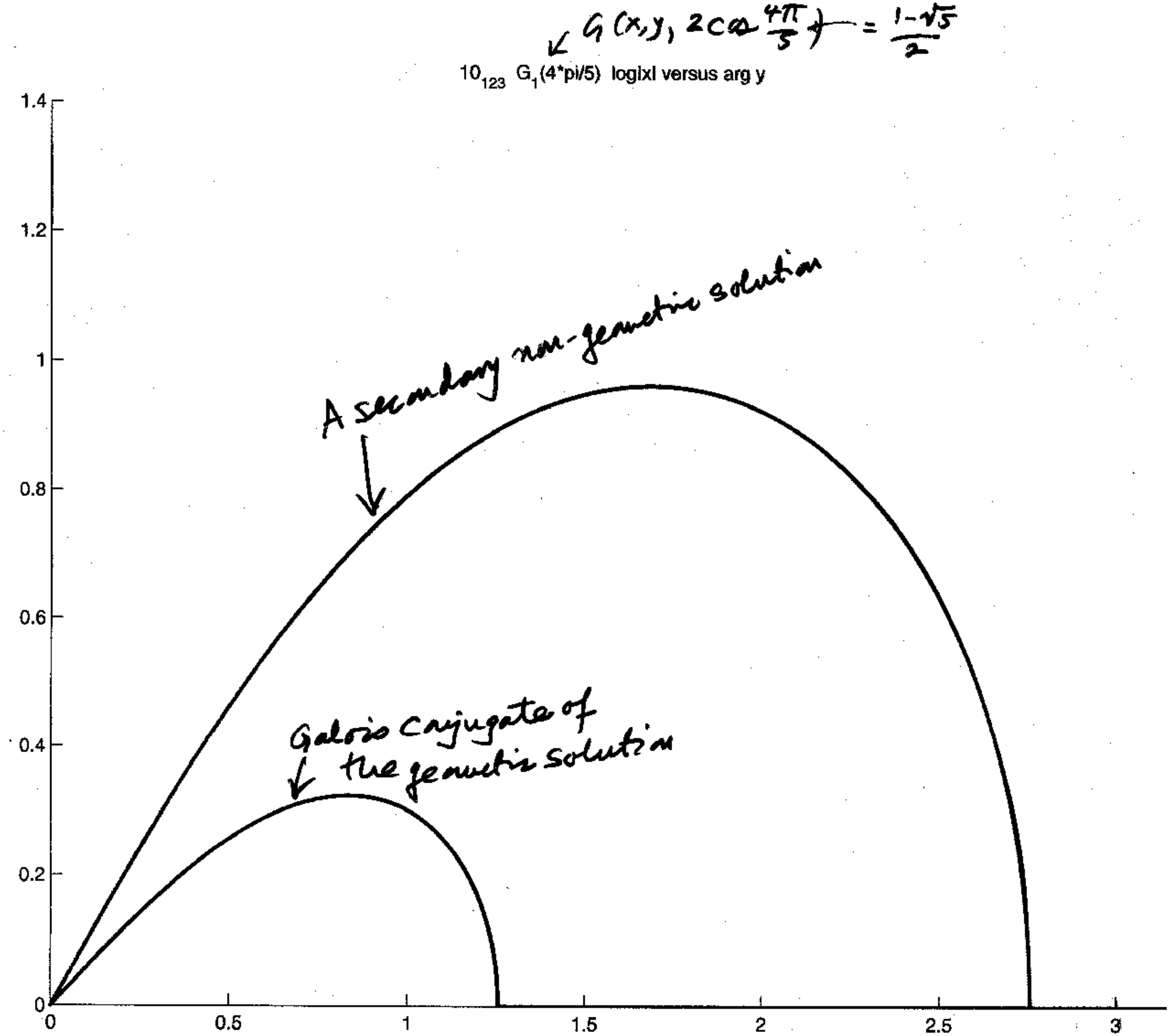
The "true" Apoly of 10^*

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & 0 & 0 \\
0 & 0 & -\frac{55}{2}\sqrt{5}+\frac{145}{2} & 0 & 0 \\
0 & 0 & \frac{245}{2}\sqrt{5}-\frac{515}{2} & 0 & 0 \\
0 & 0 & -\frac{585}{2}\sqrt{5}+\frac{1225}{2} & 0 & 0 \\
0 & 2 & 450\sqrt{5}-1030 & 2 & 0 \\
0 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & -580\sqrt{5}+1350 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & 0 \\
0 & -\frac{25}{2}\sqrt{5}+\frac{45}{2} & 740\sqrt{5}-1620 & -\frac{25}{2}\sqrt{5}+\frac{45}{2} & 0 \\
0 & \frac{25}{2}\sqrt{5}+\frac{5}{2} & -\frac{1745}{2}\sqrt{5}+\frac{3885}{2} & \frac{25}{2}\sqrt{5}+\frac{5}{2} & 0 \\
0 & 35\sqrt{5}-75 & \frac{1915}{2}\sqrt{5}-\frac{4325}{2} & 35\sqrt{5}-75 & 0 \\
1 & -75\sqrt{5}+121 & -1000\sqrt{5}+2206 & -75\sqrt{5}+121 & 1 \\
0 & 35\sqrt{5}-75 & \frac{1915}{2}\sqrt{5}-\frac{4325}{2} & 35\sqrt{5}-75 & 0 \\
0 & \frac{25}{2}\sqrt{5}+\frac{5}{2} & -\frac{1745}{2}\sqrt{5}+\frac{3885}{2} & \frac{25}{2}\sqrt{5}+\frac{5}{2} & 0 \\
0 & -\frac{25}{2}\sqrt{5}+\frac{45}{2} & 740\sqrt{5}-1620 & -\frac{25}{2}\sqrt{5}+\frac{45}{2} & 0 \\
0 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & -580\sqrt{5}+1350 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & 0 \\
0 & 2 & 450\sqrt{5}-1030 & 2 & 0 \\
0 & 0 & -\frac{585}{2}\sqrt{5}+\frac{1225}{2} & 0 & 0 \\
0 & 0 & \frac{245}{2}\sqrt{5}-\frac{515}{2} & 0 & 0 \\
0 & 0 & -\frac{55}{2}\sqrt{5}+\frac{145}{2} & 0 & 0 \\
0 & 0 & \frac{5}{2}\sqrt{5}-\frac{25}{2} & 0 & 0
\end{bmatrix}
$$

knot complement 10_{123} A−poly log|x| versus arg y




G(x,y, 2cos 2π/5) = (1+√5)/2
10_123 G_1(2*pi/5) log|x| versus arg y
NB.
Changing scale by 5x
gives the picture for the true
A-poly, A_1
πm(A_1) = vol
geometric solution
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3



G(x, y, 2cos 4π/5) = (1-√5)/2
10_123 G_1(4*pi/5) log|x| versus arg y
A secondary non-geometric solution
Galois conjugate of the geometric solution
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3

For dodec$_1$, the corresponding link is the 2-bridge link $8_8^2 = [2\ 1\ 1\ 1\ 1\ 2] = X$

The Gpoly is $16 \times 16 \times 12$ in x, y, w
height 2402

The Apoly of dodec$_1$ = X_5
is of degree 32×160

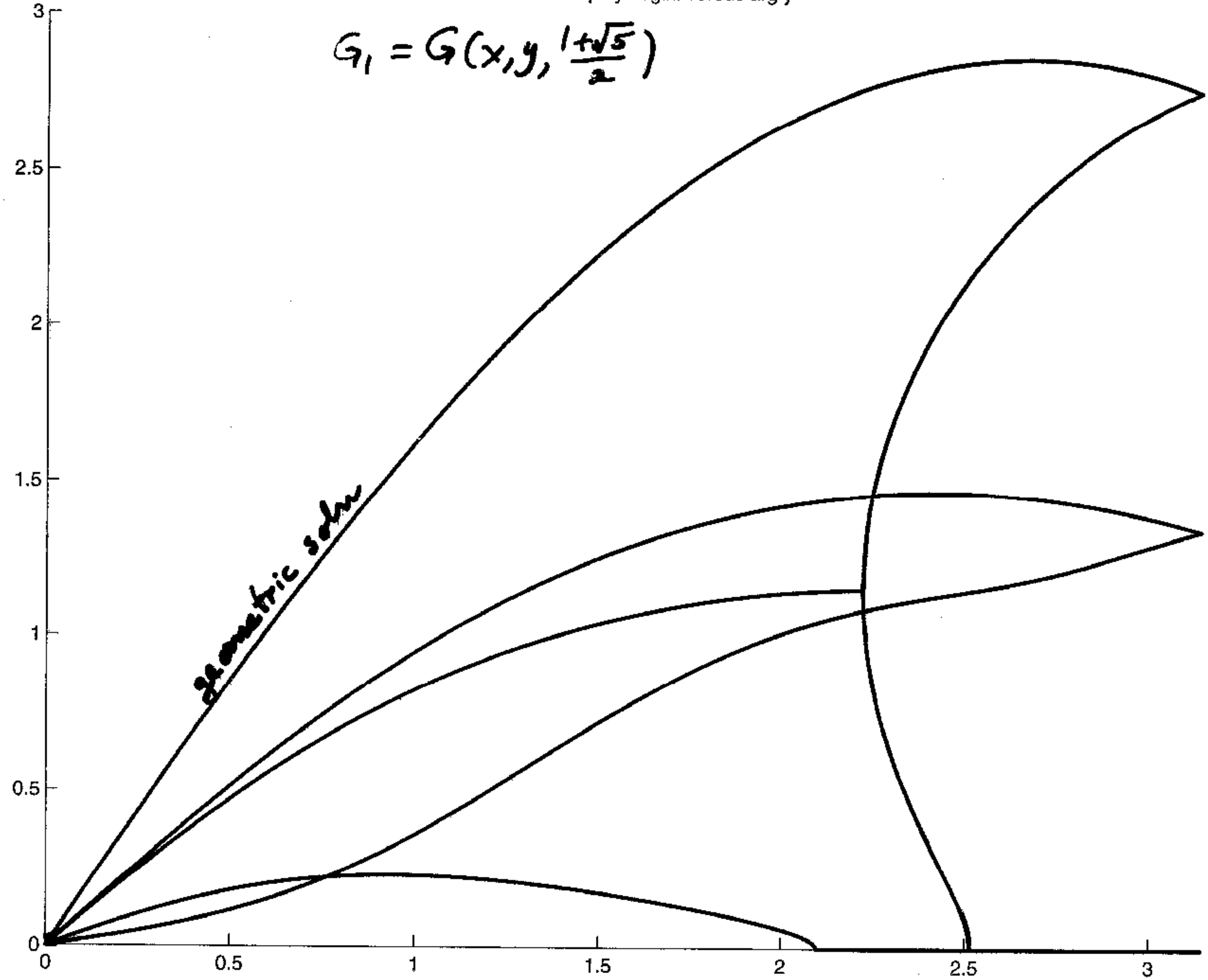
and of height
40233375155685120881065697593844

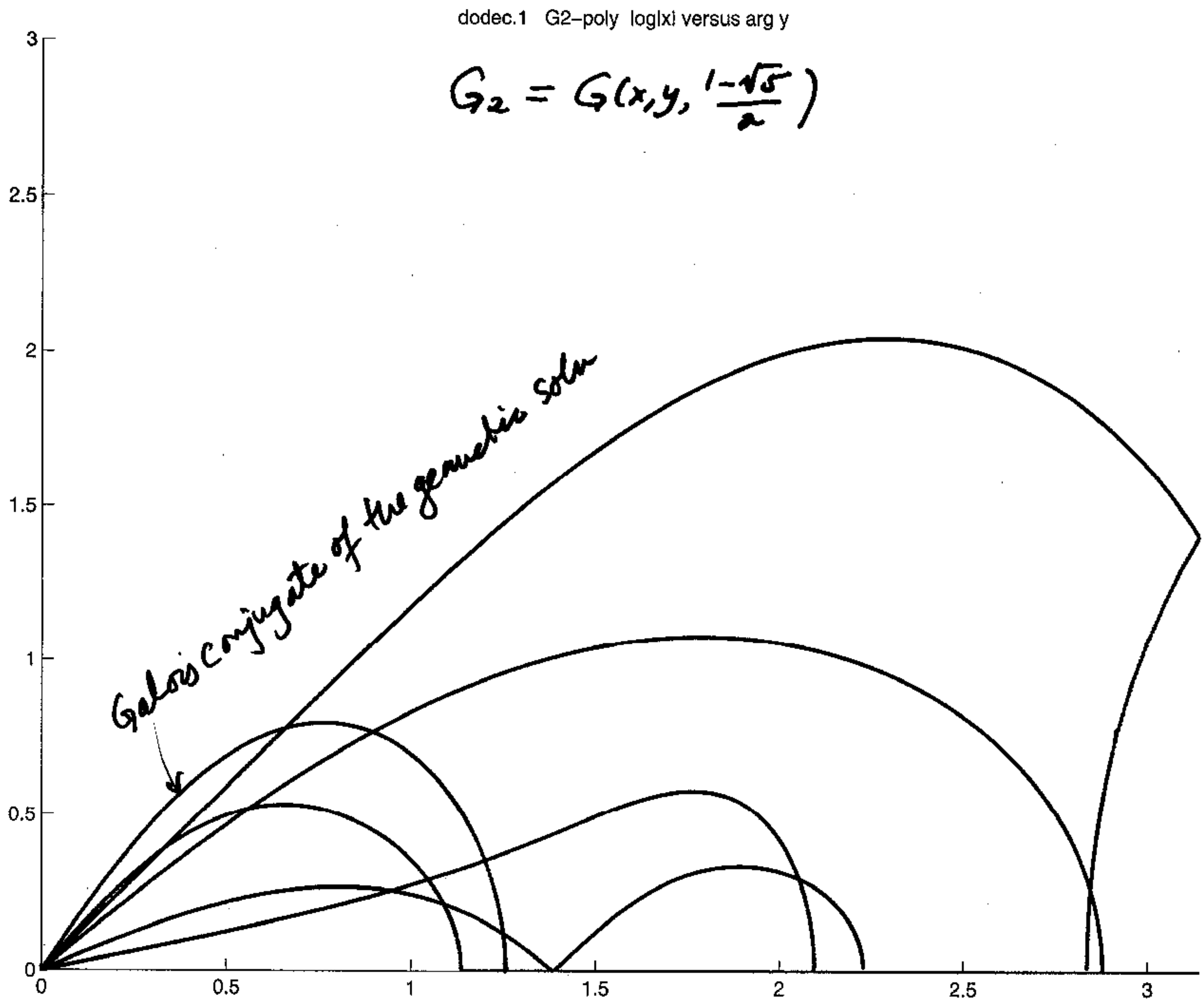
$A(x^5, y)$ = product of 10 polys of deg 16×16
in $\mathbb{Q}(\zeta_{10})[x,y]$

Clearly not accessible to the earlier methods.

The "true" Apoly, A_1 is of degree 16×80 and
height $1242362331216058526094 8$
$+ 555601325056832230774 00\sqrt{5}$

dodec.1 G1–poly log|x| versus arg y





dodec.1 G2−poly log|x| versus arg y
$G_2 = G(x, y, \frac{1-\sqrt{5}}{2})$
Galois conjugate of the generic soln
0
0.5
1
1.5
2
2.5
3

a D_9 example

(Hoste, Thistlethwaite & Weeks)

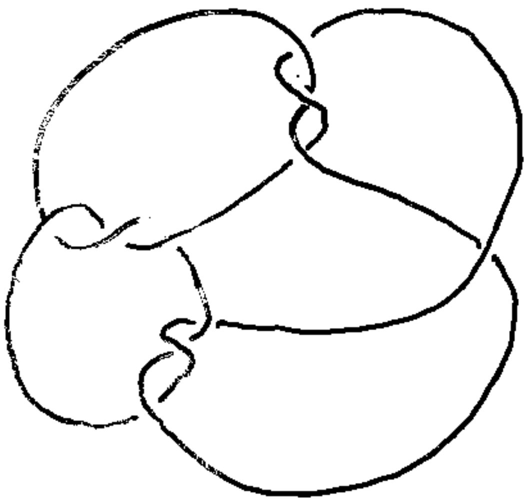
HTW found a unique example of a knot with ≤ 16 crossings with $Sym = D_9$ namely $16n1007813 =: K9$

(See Thistlethwaite's web page for beautiful pictures of this knot)

It has Inv: $3, [1,1], -31$

& $Vol = 23.747499$

In fact $K9 = (L10a119)_3$ where $L10a119 = L$ is:



$L = K(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$

a Montesinos link with $Sym \cong D_6$,

triangulation: $12 [\sqrt{-1}]$

An arithmetic manifold.

To compute $G(x,y,w)$ for L we observe that L is itself a 3-cover of the 2-cusped manifold $m125$ (Not a link complement)

The G poly for m125 is 6 × 3 × 1

$$G_0(x,y,w) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & -w & 1 & 2w & -3 & -w & \cdot \\ \cdot & w & 3 & -2w & -1 & w & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

If $G_1 = \mathrm{Res}(G_0(xz,y,w), z^3-1, z)\big|_{x=x^{1/3}}$ 6 × 9 × 3

then $G = y^8 G_1(xy^{-2}, y, w)$ 6 × 13 × 3

is the G poly for L

Then using G to compute the A poly of K9

we find $A(x,y)$ 6 × 39 with

$$A(x,1) = (x+1)^3 (x-1)^3$$

Surprisingly the shape field for both solns from $(x,y) = (-1,1)$ (the geometric soln) and $(x,y) = (1,1)$ (a 2nd soln) is the same i.e. 3, [1,1], −31 and we find

$$\pi\, m(A) = \frac{6}{5}\, \mathrm{vol}(K9)$$

a D_7 example

The unique knot with Sym $= D_7$ in HTW is $K_7 := 16n1008298$ with Vol $= 26.1598042312$, 29 tetrahedra.

We observe that

$$\frac{1}{7} \mathrm{Vol} = 3.7371148\ldots$$
$$= \mathrm{Vol}(v0160)$$

where v0160 (not a knot complement) has Sym $= \mathbb{Z}_2$ and a unique cyclic 7-cover which SnapPea verifies is $\cong K_7$

Computing the Apoly of v0160 (from π_1) we obtain A_0 12×11 ht 14

& then $A_7 = \mathrm{Res}(A_0(xz, y), z^7 - 1, z)\Big|_{x = x^{1/7}}$

and finally

$$A(x,y) = y^{-40} A_7(x y^8, y) \quad 12 \times 93$$

ht 1889695362

is the Apoly of K_7.

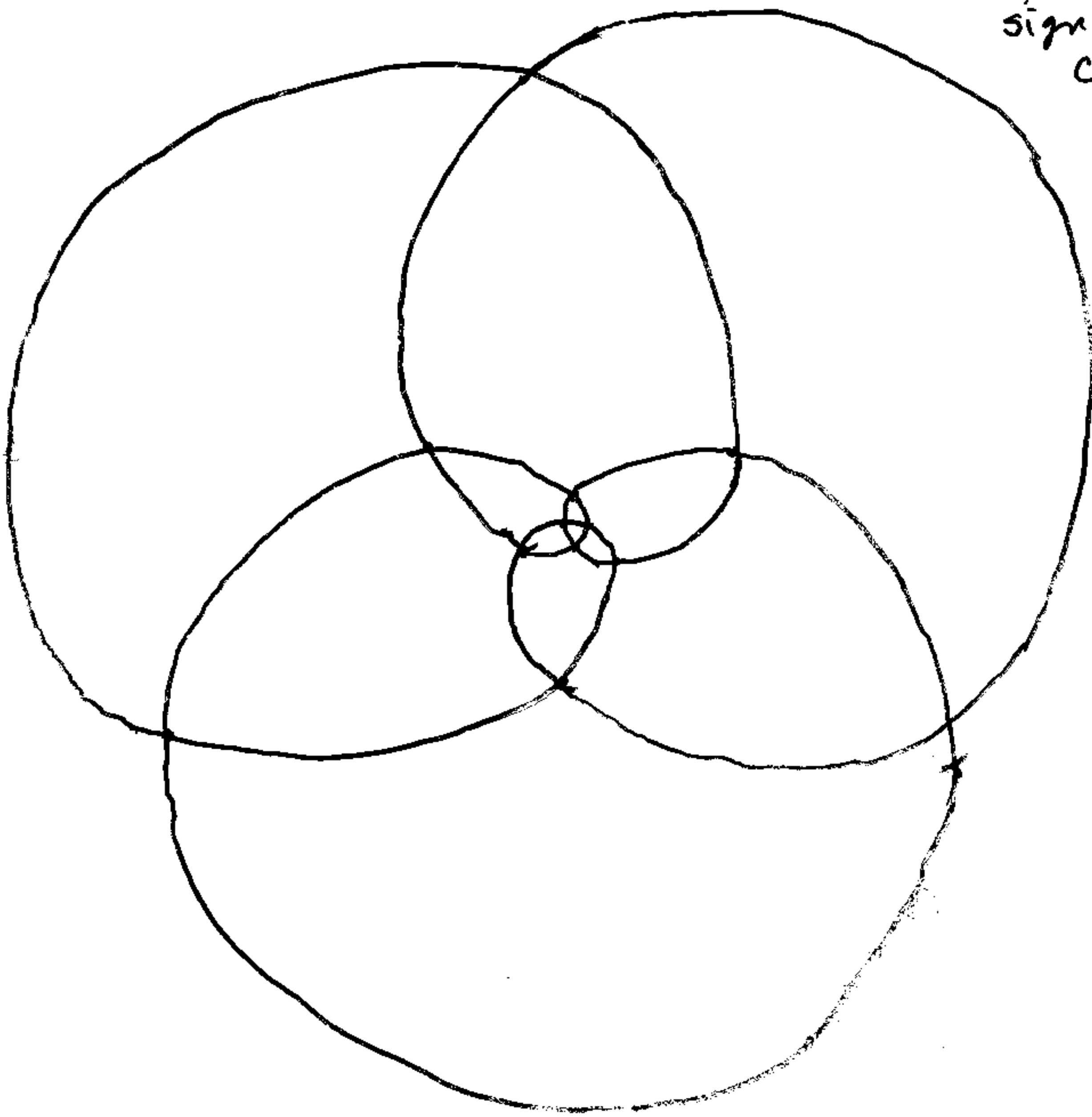
N.B. Here there is no branched cover, just a cyclic 7-cover

12n706 et al

$12n706 = [55]_3$ D_6 amphicheiral

(or $Th(3,5) = [3223]_3$, $[343]_3$, $[523]_3$, $[73]_3$ ← depending on the sign of the crossings

or $Torus(3,5) = [10]_3$)



12n706 is one of 3 known knots with Cu < Sh

It has $vol = 13.41737439 = D(8[\sqrt{-1}] + 6[\frac{1+\sqrt{-3}}{2}])$

$A(x,y)$ computed from the Gpoly of [55]

is 12×42 ht 60129

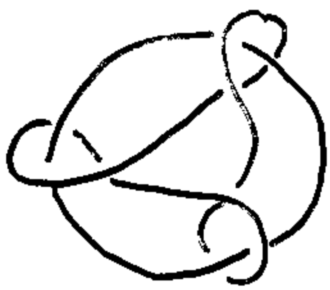
Links with > 2 components

e.g. $X = 6_1^3 =$  3 link alternating chain with Sym $= D_6$

$= 2,2,2$

X_n is a 2-link with Sym $= D_{2n}$ $= K(\underbrace{\tfrac{2}{3}, \tfrac{2}{3}, \dots, \tfrac{2}{3}}_{n})$

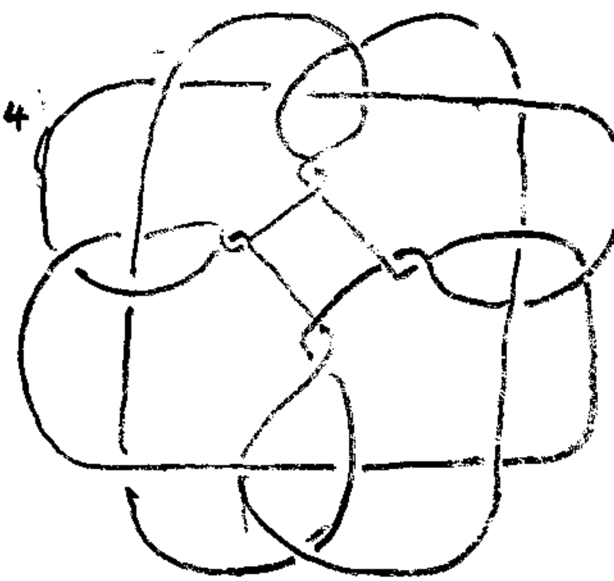
$X_{n,m}$ is a knot with Sym $= D_{mn}$ if $(m,n) = 1$

e.g. $X_3 =$ 

Sym D_6

$X_{3,4}$

Sym D_{12}



for $X_{3,4}$ A is 6×36 ht 8272224360

$A(x^{12}, y)$ is 72×36 is a product of 12 $H(\zeta_{mn} x, y, \omega_{2m}, \omega_{2n})$

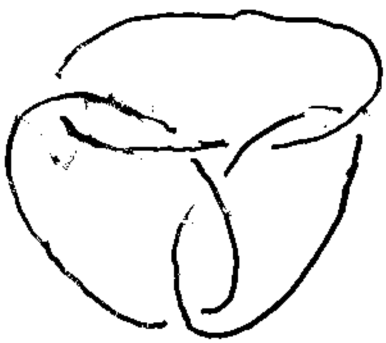
$$H = \begin{bmatrix} & & 1 & -vw & -2+v^2+w^2 & -vw & 1 \\ & & -v^2-w^2 & 5vw & 2v^2+2w^2 & vw & \\ & -vw & 2v^2+2w^2 & -5vw & v^2+w^2 & & \\ -1 & vw & 2-v^2-w^2 & vw & -1 & & \end{bmatrix}$$

$H(x,y;v,w)$ degree $6 \times 3 \times 2 \times 2$

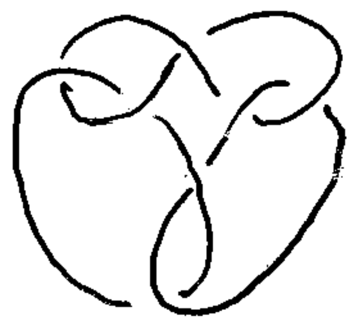
For example this gives the polynomials for the sequence

$$X_{n,2} = X_{2,n} = K(\underbrace{\tfrac{1}{3}, \ldots, \tfrac{1}{3}}_{n}) \quad n \text{ odd.}$$

The sequence of 3-links



2, 2, 2



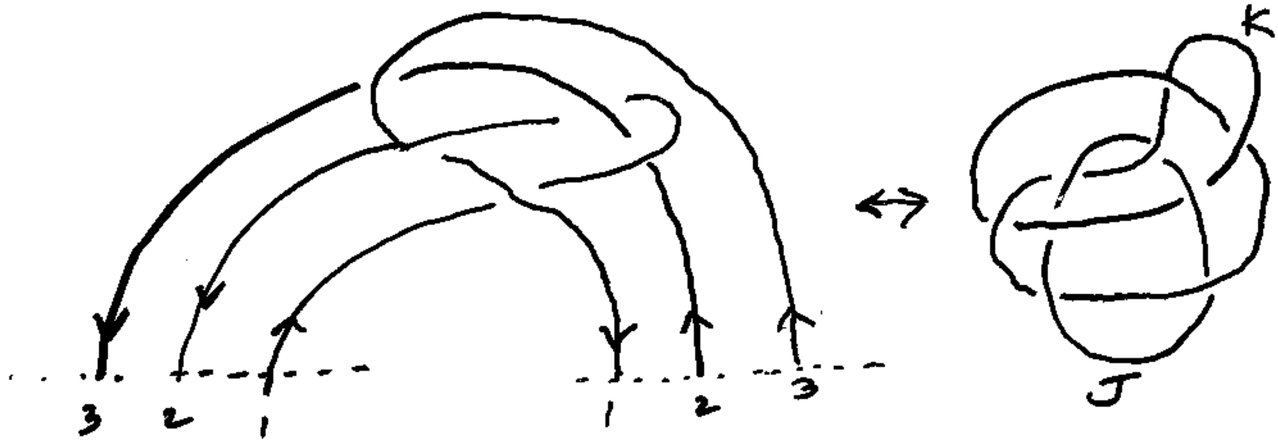
2, 2, 2+ . . . .

can be handled in a similar way, e.g. for 2, 2, 2+ $H(x, y, v, w)$ is $10 \times 5 \times 4 \times 4$

Then $X_{2,n} = K(\underbrace{\tfrac{1}{5}, \tfrac{1}{5}, \ldots, \tfrac{1}{5}}_{n \text{ times}})$ n odd.

Can all X_n be knots?

e.g. template for L9a31 Sym = trivial, chiral



Exercise: Show

All X_n are knots with $Sym(X_n) = Z_n$

$\#crossings(X_n) = 5n$

e.g. $X_2 = 10_{98}$ (= 10a 96), $X_3 = 15a82698$, ... (the unique Z_3 in HTW)

Challenge:

$G(x,y,w)$ is $24 \times 24 \times ?$ in x, y, w

— not yet computed

Exercise: If $\Delta_n(y)$ is the Alexander poly of X_n, show $(y^2-y+1)^n \mid \Delta_n(y)$