PERFECT POWERS WITH FEW TERNARY DIGITS

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Abstract

We classify all integer squares (and, more generally, q-th powers for certain values of q) whose ternary expansions contain at most three digits. Our results follow from Padé approximants to the binomial function, considered 3-adically.

–Dedicated to the memory of John Selfridge.

1. Introduction

If we fix an integer base $b > 1$ and let $B_k(b)$ denote the set of integers whose base b representation contains at most k nonzero "digits", then standard density arguments suggest that for a typical sequence S of positive integers, with suitable growth rate, the intersection $S \cap B_k(b)$ should be a finite set. Quantifying this statement for any given S can be remarkably difficult. In the case where S consists of the positive integer squares, then $S \cap B_3(b)$ is not actually finite (as the identity $(1 + b^\ell)^2 = 1 + 2b^\ell + b^{2\ell}$ for $\ell \geq 1$ reveals), yet a result of Corvaja and Zannier [5] implies that all but finitely many squares in $B_3(b)$ can be classified by means of such polynomial identities. The proof of this result in [5], however, depends upon Schmidt's Subspace Theorem and is thus ineffective (in that it does not allow one to determine the implicit exceptional set). Analogous questions for $B_4(b)$ appear to be almost completely open (but see [6] in case $b = 2$).

Szalay [8] employed rather different means to deduce a complete classification of odd squares with three binary digits. He proved the following.

Theorem S If y is an odd positive integer such that y^2 has at most three binary digits, then $y = 7$, $y = 23$ or $y = 2^t + 1$ for some positive integer t.

The arguments of [8], which rely on a result of Beukers based upon Padé approximation, do not appear to readily extend to bases $b > 2$. In this short note,

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however, we will employ a somewhat different approach to treat the case $b = 3$. We prove

Theorem 1 *If y is a positive integer with, say, y coprime to 3, then if y^2 has at most three ternary digits, it follows that $y \in \{1, 5, 8, 13\}$ or $y = 3^t + 1$, where t is a nonnegative integer.*

Note that the squares y^2 with three ternary digits which are divisible by 3 may be obtained from the values listed here via multiplication by a suitable power of 3. In a recent paper of Bugeaud, Mignotte and the author [2], the result of Szalay is extended to higher powers y^q for $q > 2$. The techniques of [2] do not apparently provide an absolute upper bound upon q for which y^q has at most three ternary digits (though they do precisely this under the assumption that $y \equiv 1 \pmod 3$). It is, however, possible to prove the following.

Theorem 2 *If y is a positive integer with y coprime to 3, then if y^q has at most three ternary digits for $q = 3$ or $7 \leq q < 1000$ prime, it follows that $(y, q) = (13, 3)$.*

Observe that we make no claims regarding the case $q = 5$. Indeed, we are unable to effectively solve the equation

$$3^a + 3^b + 2 = y^5.$$

Presumably, it has no solutions in positive integers a and b with $a > b$, other than $(a, b) = (3, 1)$. By the main theorem of [5], for each value of q excluded in the above theorem (i.e. $q = 3$ and prime $q > 1000$), there are at most finitely many qth powers with at most three ternary digits, though the result is ineffective.

At the risk of being accused of trying to impart Theorems 1 and 2 with undue significance, one might mention that they represent effective solutions of (very simple) cases of a deep conjecture of Lang and Vojta on the Zariski denseness of S-integral points on certain algebraic varieties (see e.g. page 486 of [7]).

2. Squares With 3 Ternary Digits

We begin by considering the case of squares with at most 2 ternary digits. These correspond, assuming that $\gcd(3, y) = 1$ and $y > 1$, to the Diophantine equation

$$2^{\delta_1} 3^a + 2^{\delta_2} = y^2,$$

where $\delta_i \in \{0, 1\}$ and $a > 0$. Modulo 12, it follows that $\delta_1 = \delta_2 = 0$ and so, after factoring $y^2 - 1$, we have that $a = 1$ and $y = 2$.

We now turn our attention to squares with 3 ternary digits. A priori, if we suppose that y is coprime to 3, we are led to the Diophantine equation

$$2^{\delta_1}3^a + 2^{\delta_2}3^b + 2^{\delta_3} = y^2, \tag{1}$$

where $\delta_i \in \{0,1\}$ and $a > b > 0$. Modulo 3, however, and crucially for our argument, we may suppose that $\delta_3 = 0$. Modulo 8, we may also assume that $(\delta_1, \delta_2) \neq (0,0)$. To simplify matters, we check that there are no unexpected solutions with $1 \leq b < a \leq 200$; we may thus suppose that $a > 200$. Our argument proceeds as follows. Firstly, we will construct off-diagonal Padé approximants to $(1+x)^{1/2}$ and use these to show that solutions to equation (1) necessarily have $a < 16b$. From this, we will deduce a contradiction via local arguments which force a to be substantially larger than b. It is worth observing that the result of Beukers which is key to Theorem S also appeals to Padé approximants to $(1+x)^{1/2}$ in order to derive a lower bound upon the quantity $|2^a - y^2|$. The key difference is that the values of x are chosen to be small in Archimedean terms, while we will be considering x which are small 3-adically (and indeed large in Archimedean absolute value). Such an approach is already present in another paper of Beukers [4] and, more recently, in work of Corvaja and Zannier [6]; our argument closely follows the latter.

We begin by writing down the Padé approximants to $(1+x)^{1/2}$. Specifically, if n_1 and n_2 are positive integers, define

$$P_{n_1,n_2}(x) = \sum_{k=0}^{n_1} \binom{n_2+1/2}{k}\binom{n_1+n_2-k}{n_2} x^k \tag{2}$$

and

$$Q_{n_1,n_2}(x) = \sum_{k=0}^{n_2} \binom{n_1-1/2}{k}\binom{n_1+n_2-k}{n_1} x^k. \tag{3}$$

Then, as in [1], we have that

$$P_{n_1,n_2}(x) - (1+x)^{1/2}\ Q_{n_1,n_2}(x) = x^{n_1+n_2+1} E_{n_1,n_2}(x), \tag{4}$$

where (see e.g. Beukers [3])

$$E_{n_1,n_2}(x) = \frac{(-1)^{n_2}\,\Gamma(n_2+3/2)}{\Gamma(-n_1+1/2)\Gamma(n_1+n_2+1)} F(n_1+1/2, n_1+1, n_1+n_2+2, -x), \tag{5}$$

for F the hypergeometric function given by

$$F(a,b,c,-x) = 1 - \frac{a\cdot b}{1\cdot c}x + \frac{a\cdot(a+1)\cdot b\cdot(b+1)}{1\cdot 2\cdot c\cdot(c+1)}x^2 - \cdots .$$

Appealing twice to (4) and (5) with, in the second instance, n_1 replaced by n_1+1, and eliminating $(1+x)^{1/2}$, we find that $P_{n_1+1,n_2}(x)Q_{n_1,n_2}(x) - P_{n_1,n_2}(x)Q_{n_1+1,n_2}(x)$

is a polynomial of degree n_1+n_2+1 with a zero at $x=0$ of order n_1+n_2+1 (and hence is a monomial). It follows that we may write

$$P_{n_1+1,n_2}(x)Q_{n_1,n_2}(x)-P_{n_1,n_2}(x)Q_{n_1+1,n_2}(x)=cx^{n_1+n_2+1}. \tag{6}$$

Here, we may show that

$$c=(-1)^{n_2+1}\frac{(n_1+n_2+1)\Gamma(n_2+3/2)}{(n_1+1)!\,n_2!\,\Gamma(-n_1+1/2)}.$$

The precise value of the constant c here is unimportant for our purposes; it is enough to note that it is nonzero. We choose $n_2=\lceil a/4b\rceil$, i.e. the smallest integer not less than $a/4b$, and let $n_1=3n_2-\delta$ for one of $\delta\in\{0,1\}$. It is useful for us to observe that

$$\binom{n+\frac{1}{2}}{k}4^k\in\mathbb{Z},$$

so that, in particular, $4^{n_1}P_{n_1,n_2}(x)$ and $4^{n_1}Q_{n_1,n_2}(x)$ are polynomials with integer coefficients.

Setting $\eta=\sqrt{1+2^{\delta_2}3^b}$, since $(1+x)^{1/2}$, $P_{n_1,n_2}(x)$ and $Q_{n_1,n_2}(x)$ have 3-adic integral coefficients, the same is necessarily true of $E_{n_1,n_2}(x)$ and so, via equation (4),

$$\left|4^{n_1}P_{n_1,n_2}(2^{\delta_2}3^b)-\eta\,4^{n_1}Q_{n_1,n_2}(2^{\delta_2}3^b)\right|_3\leq 3^{-a}.$$

On the other hand, from the fact that $\eta^2\equiv y^2 \pmod{3^a}$, we have

$$\eta\equiv(-1)^\kappa y \pmod{3^a},$$

for some $\kappa\in\{0,1\}$, and hence

$$\left|4^{n_1}P_{n_1,n_2}(2^{\delta_2}3^b)-(-1)^\kappa y\,4^{n_1}Q_{n_1,n_2}(2^{\delta_2}3^b)\right|_3\leq 3^{-a}.$$

Equation (6) readily implies that for at least one of $\delta\in\{0,1\}$, we must have

$$P_{n_1,n_2}(2^{\delta_2}3^b)\neq(-1)^\kappa y\,Q_{n_1,n_2}(2^{\delta_2}3^b)$$

and hence, for the corresponding choice of n_1,

$$\left|4^{n_1}\,P_{n_1,n_2}(2^{\delta_2}3^b)-(-1)^\kappa y\,4^{n_1}\,Q_{n_1,n_2}(2^{\delta_2}3^b)\right|\geq 3^a. \tag{7}$$

From (2) and (3), after some relatively routine calculus, we may conclude that

$$\left|4^{n_1}\,P_{n_1,n_2}(2^{\delta_2}3^b)\right|<(n_1+1)\left|\binom{n_2+\frac{1}{2}}{n_1}\right|(8\cdot 3^b)^{n_1}<5^{n_2}\,3^{bn_1}$$

and

$$\left|4^{n_1}\,Q_{n_1,n_2}(2^{\delta_2}3^b)\right|<(n_2+1)\binom{n_1-\frac{1}{2}}{n_2}(2\cdot 3^b)^{n_2}\,4^{n_1}<7^{n_2}\,3^{bn_2},$$

whereby, from $|y| < 2^{1/2} \cdot 3^{a/2}$ and (7),

$$3^a < 5^{n_2}\, 3^{bn_1} + 2^{1/2}\, 7^{n_2}\, 3^{bn_2} 3^{a/2} \leq 5^{n_2}\, 3^{3bn_2} + 2^{1/2}\, 7^{n_2}\, 3^{bn_2} 3^{a/2}.$$

Since $n_2 < 1 + a/4b$, we thus have

$$3^{a/4} < 5^{(a+4b)/4b}\, 3^{3b} + 2^{1/2}\, 7^{(a+4b)/4b}\, 3^b. \tag{8}$$

Let us assume that $a \geq 16b$. Then (8) implies that $b \leq 7$; in fact, each choice of b with $2 \leq b \leq 7$, together with (8), contradicts the further assumption that $a > 200$. In case $b = 1$, inequality (8) fails to provide such a contradiction. If $b = 1$, however, considering equation (1) modulo 8, we find that necessarily $\delta_2 = 1$ and that a is even. In case $\delta_1 = 0$, we thus have $a = 2$ and $y = 4$. If $\delta_1 = 1$, standard routines for finding integral points on models of genus one curves, applied to the quartic equations

$$y^2 = 2 \cdot 3^{2\delta} x^4 + 7,\ \delta \in \{0, 1\}$$

lead to the conclusion that $a = 2$ and $y = 5$.

It remains, then, to handle the situation where $a < 16b$. We will appeal to straightforward local arguments, providing full details for $(\delta_1, \delta_2) = (0, 1)$; the cases $(\delta_1, \delta_2) = (1, 0)$ and $(1, 1)$ are essentially similar.

Suppose then that $(\delta_1, \delta_2) = (0, 1)$ and that we have a solution to equation (1). Since $\nu_3(y^2 - 1) = b$, it follows that either $y = 3^b - 1$, $y = 3^b + 1$, or $y \geq 5 \cdot 3^b - 1$. In the first case, we have

$$3^a + 2 \cdot 3^b + 1 = 3^{2b} - 2 \cdot 3^b + 1$$

and so $3^a = 3^{2b} - 4 \cdot 3^b$, whereby $b = a$, a contradiction. The second case leads to our infinite family with $a = 2b$. We may therefore suppose that $y \geq 5 \cdot 3^b - 1$ and thus $a \geq 2b + 3$. Considering the Taylor series

$$(1 + x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \cdots, \tag{9}$$

and viewing $x = 3^a + 2 \cdot 3^b$ as a 3-adic integer, we have, from $a \geq 2b + 3$, that

$$\nu_3(y \pm (1 + 3^b)) \geq 2b.$$

We thus have

$$y \geq 3^{2b} - 3^b - 1$$

and so, after a little work, $a \geq 4b$. Again considering (9), we now find that

$$\nu_3(y \pm (1 + 3^b - 3^{2b}/2)) = 3b$$

and so

$$y \geq 3^{3b} - 3^{2b}/2 + 3^b + 1.$$

Again appealing to the equality $3^a + 2 \cdot 3^b + 1 = y^2$, we deduce, after a short argument, that $a \geq 6b$. Continuing in this vein,

$$\nu_3 \left(y \pm \left(1 + 3^b - \frac{3^{2b}}{2} + \frac{3^{3b}}{2} - \frac{5 \cdot 3^{4b}}{8} + \frac{7 \cdot 3^{5b}}{8} \right) \right) = 6b,$$

whence

$$y \geq 3^{6b} - \frac{7 \cdot 3^{5b}}{8} + \frac{5 \cdot 3^{4b}}{8} - \frac{3^{3b}}{2} + \frac{3^{2b}}{2} - 3^b + 1$$

and $a \geq 12b$. Finally, we have

$$\nu_3 \left(y \pm \left(1 + 3^b - \frac{3^{2b}}{2} + \frac{3^{3b}}{2} - \frac{5 \cdot 3^{4b}}{8} + \frac{7 \cdot 3^{5b}}{8} - \frac{21 \cdot 3^{6b}}{16} + \frac{33 \cdot 3^{7b}}{16} \right) \right) = 8b$$

and so

$$y \geq 3^{8b} - \frac{33 \cdot 3^{7b}}{16} + \frac{21 \cdot 3^{6b}}{16} - \frac{7 \cdot 3^{5b}}{8} + \frac{5 \cdot 3^{4b}}{8} - \frac{3^{3b}}{2} + \frac{3^{2b}}{2} - 3^b + 1. \tag{10}$$

Since we assume that $a > 200$ and $a < 16b$, it follows that $b > 12$. Combining (10) with the equation $3^a + 2 \cdot 3^b + 1 = y^2$ implies that $a \geq 16b$. The resulting contradiction enables us to conclude as desired.

3. Higher Powers With 3 Ternary Digits

In this section, we will prove Theorem 2. The (great) majority of the work here was already done in [2], where we find

Theorem 3 *If there exist integers $a > b > 0$ and $q \geq 2$ for which*

$$x^a + x^b + 1 = y^q, \quad \text{with} \quad x \in \{2, 3\},$$

then (x, a, b, y, q) is one of

$(2, 5, 4, 7, 2)$, $(2, 9, 4, 23, 2)$, $(3, 7, 2, 13, 3)$, $(2, 6, 4, 3, 4)$, $(4, 3, 2, 9, 2)$ *or* $(4, 3, 2, 3, 4)$,

or $(x, a, b, y, q) = (2, 2t, t + 1, 2^t + 1, 2)$, for some integer $t = 2$ or $t \geq 4$.

In particular, it remains only to solve the equation

$$2^{\delta_1} 3^a + 2^{\delta_2} 3^b + 2^{\delta_3} = y^q, \tag{11}$$

where $(\delta_1, \delta_2, \delta_3) \neq (0, 0, 0)$, $a > b > 0$ and $q = 3$ or $7 \leq q < 1000$ is prime. In each case under consideration, it is a routine (if not especially fast) matter to find local obstructions to (11); i.e. to find N such that the equation in insoluble modulo N.

We construct our values N as products of certain primes $p_i \equiv 1 \pmod{q}$ for which $\text{ord}_2(p_i) = mq$ with m a relatively small integer. Here, $\text{ord}_l(p_i)$ denotes the smallest positive integer k for which $l^k \equiv 1 \pmod{p_i}$. Fixing an integer M, for each such p_i with $m \mid M$, we let a and b loop over integers from 1 to Mq and store the resulting pairs (a, b) with the property that either $2^{\delta_1}3^a + 2^{\delta_2}3^b + 2^{\delta_3} \equiv 0 \pmod{p_i}$ or

$$\left(2^{\delta_1}3^a + 2^{\delta_2}3^b + 2^{\delta_3}\right)^{(p_i-1)/q} \equiv 1 \pmod{p_i}.$$

For a given prime p_i, if we denote by S_i the set of corresponding pairs (a, b), then we wish to find M and corresponding primes $p_1, p_2, \ldots, p_k$ for which

$$\bigcap_{i=1}^{k} S_i = \emptyset. \tag{12}$$

We check that such sets of primes exist (with M reasonably small) for each prime $q = 3$ or $5 \leq q < 1000$, and each triple $(\delta_1, \delta_2, \delta_3)$. By way of example, if we consider equation (11) in case $q = 439$ and $(\delta_1, \delta_2, \delta_3) = (0, 0, 1)$, we may take $M = 1440$ and $p_i \in \{4391, 13171, 39511, 70241, 105361\}$. Full details and the Maple code used for these computations are available from the author on request.

If $q = 5$, it is easy, as in other cases, to find local obstructions, provided $(\delta_1, \delta_2, \delta_3) \neq (0, 0, 1)$. In the situation where $(\delta_1, \delta_2, \delta_3) = (0, 0, 1)$, the solution with $(a, b) = (3, 1)$ ensures the failure of such a simple approach.

4. Concluding Remarks

The arguments of this paper are apparently not sufficient to prove like results for bases $b > 4$. The principal reason for this is that they rely upon the assumption that the given power y^q which one wishes to conclude to have at least, say, 4 digits in base b, satisfies $y^q \equiv 1 \pmod{b}$. Such a supposition is essentially without loss of generality only for $b = 2$ or 3.

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