

# RAMANUJAN-NAGELL CUBICS

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ABSTRACT. A well-known result of Beukers [3] on the generalized Ramanujan-Nagell equation has, at its heart, a lower bound on the quantity  $|x^2 - 2^n|$ . In this paper, we derive an inequality of the shape  $|x^3 - 2^n| \geq x^{4/3}$ , valid provided  $x^3 \neq 2^n$  and  $(x, n) \neq (5, 7)$ , and then discuss its implications for a variety of Diophantine problems.

**1. Introduction.** Surfing the internet one day, the second author came across a conversation on a physics forum [10], in which a Diophantine problem was proposed. The proposer wished to find a proof of his conjecture, to the effect that the Diophantine equation

$$(1) \quad x^3 - x + 8 = 2^k$$

has no solutions in integers  $x > 8$ . Since it indeed has solutions with

$$x \in \{-2, -1, 0, 1, 3, 5, 8\},$$

such a result would be, in some sense, best possible.

That equations such as (1) have at most finitely many solutions is an immediate consequence of a classical result of Siegel [12], and, in fact, if we denote by  $P(m)$  the greatest prime factor of a nonzero integer  $m$ , one may show (see e.g. [7]) that

$$(2) \quad P(f(x)) \geq c_1 \cdot \log \log \max\{|x|, 3\}.$$

Here  $c_1 = c_1(f) > 0$  and  $f$  is, say, an irreducible polynomial with integer coefficients and degree at least two. Even more, if  $F(x, y)$  is an irreducible binary form (i.e. homogeneous polynomial) with integer coefficients and degree at least 3, work of Mahler [8], as extended by

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Shorey et al [11], implies that

$$(3) \quad P(F(x, y)) \geq c_2 \cdot \log \log \max\{|x|, |y|, 3\},$$

with  $c_2 = c_2(F) > 0$ , so that, given primes  $p_1, p_2, \dots, p_n$ , the *Thue-Mahler* equation

$$(4) \quad F(x, y) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$$

has at most finitely many solutions in coprime integers  $x$  and  $y$ , and nonnegative integers  $\alpha_1, \dots, \alpha_n$ . In particular, the equation

$$x^3 - xy^2 + 8y^3 = 2^k$$

can be satisfied by at most finitely many coprime integers  $x$  and  $y$ , and nonnegative  $k$  – those with  $y = 1$  provide the solutions to (1). Statements (2) and (3) can both be made effective; the interested reader is directed to [13] for details.

The motivation for studying such an equation in [10] was, apparently, to find a cubic analog of the Ramanujan-Nagell equation  $x^2 + 7 = 2^k$ , which is known (see e.g. [9]) to have precisely the integer solutions corresponding to  $|x| = 1, 3, 5, 11$  and 181. This is extremal in the sense that there exists no monic quadratic  $f(x)$  for which  $f(x) = 2^k$  has more than ten solutions in integers  $x$ , via the following theorem of Beukers [3].

**Theorem 1.** (*Beukers*) *If  $D$  is an odd integer, then the equation*

$$x^2 + D = 2^n$$

*has at most 5 solutions in positive integers  $x$ .*

The only monic irreducible cubics  $f$  we know for which the equation  $f(x) = 2^k$  has more solutions than the seven to (1) are those corresponding to the polynomial  $f(x) = x^3 - 13x + 20$  and its translates, each with 8 solutions. The results of this paper make it a routine matter to solve such an equation for any monic cubic (and the machinery we employ readily generalizes to certain non-monic cases, though we omit the details in the interest of keeping our exposition reasonably simple). An example of what we prove is the following.

**Proposition 2.** *Let  $b$  and  $c$  be integers, and suppose that  $x$  and  $k$  are integers for which*

$$(5) \quad |x^3 + bx + c| = 2^k.$$

*Then either  $b = c = 0$ , or  $x = -c/b$ , or  $(x, b, c, k) = (5, 0, 3, 7)$ ,  $(-5, 0, -3, 7)$ , or we have*

$$(6) \quad |x| \leq \max \left\{ |2b|^3, |2c|^{3/4} \right\}.$$

Applying this to equation (1) implies that its solutions satisfy  $|x| \leq 8$ , whereby a routine check leads to the desired conclusion.

Our real motivation in writing this paper is to emphasize the unnaturally large influence a single numerical “fluke” can have upon certain results in explicit Diophantine approximation. Underlying Theorem 1 of Beukers is an inequality of the shape  $|x^2 - 2^n| \gg 2^{0.1n}$ , valid for odd  $n$  and derived through Padé approximation, through appeal to the identity  $181^2 + 7 = 2^{15}$  (which implies that  $|\sqrt{2} - \frac{181}{2^7}|$  is “small”). Proposition 2 follows from a rather similar approach and depends fundamentally upon the relation  $5^3 + 3 = 2^7$  (which implies that  $|\sqrt[3]{2} - \frac{5}{2^2}|$  is also “small”). The fact that we are able to prove an effective inequality of the shape

$$\left| \sqrt[3]{2^r} - \frac{p}{2^k} \right| \gg 2^{-\lambda k},$$

for  $r \in \{1, 2\}$  and, crucially,  $\lambda < 2$ , is what enables us to derive results like Proposition 2. It is worth observing that techniques based upon lower bounds for linear forms in logarithms lead to upper bounds for the heights of solutions to much more general equations than (5), but have the disadvantage of these bounds being significantly worse than exponential in the coefficients  $b$  and  $c$ .

The outline of this paper is as follows. In Section 2, we begin by stating our results, expressed both in terms of explicit rational approximation to certain algebraic numbers by rational with restricted denominators, and also as results about corresponding Diophantine equations. In Sections 3 and 4, we collect the necessary technical lemmata about Padé approximation to binomial functions (at least in terms of Archimedean valuations). Section 5 contains the proof of our main result, Theorem 3, modulo an arithmetic result on the coefficients of our Padé approximants, which we provide in Section 6. Sections 7,

8 and 9 consist of the proofs of Corollary 4, Theorem 5 and Theorem 6, respectively. Finally, in Section 10, we make a few remarks about more general Thue-Mahler equations.

**2. Statements of our results.** Our results are of two closely related types. First, we have an explicit lower bound for approximation to  $\sqrt[3]{2}$  or  $\sqrt[3]{4}$  by rational numbers with denominators restricted to being essentially a power of 2. From this, with what are basically elementary arguments, we are able to derive a number of results on both the heights and the number of solutions of equations of the shape  $F(x) = 2^n$ , for monic cubic polynomial  $F \in \mathbb{Z}$ . We start with the following :

**Theorem 3.** *Suppose that  $r, p, s$  and  $k$  are integers with  $r \in \{1, 2\}$ ,  $s \in \{1, 3\}$  and  $k > 12$ . Then we have*

$$(7) \quad \left| 2^{r/3} - \frac{p}{s2^k} \right| > 2^{-1.62k}.$$

As noted earlier, this “restricted irrationality measure” to  $\sqrt[3]{2}$  and  $\sqrt[3]{4}$  has a number of straightforward consequences for certain Diophantine equations. We will begin by stating an almost immediate corollary that will prove a useful form for later applications to Diophantine equations.

**Corollary 4.** *If  $x$  and  $n$  are integers with  $x^3 \neq 27 \cdot 2^n$ , then either*

$$x \in \{4, 5, 8, 15, 19, 38, 121\}$$

*or we have*

$$(8) \quad |x^3 - 27 \cdot 2^n| \geq 3^{5/3} \cdot x^{4/3}.$$

From the standpoint of explicit solution of Diophantine equations, our main result is the following (from which Proposition 2 is an immediate corollary, taking  $a = 0$ ).

**Theorem 5.** *Let  $a, b$  and  $c$  be integers, and suppose that  $x$  and  $n$  are integers for which we have*

$$(9) \quad |x^3 + ax^2 + bx + c| = 2^n.$$

Then either

- $(a, b, c) = (3t, 3t^2, t^3)$  for some integer  $t$ , or
- $(a, b, c, x, n) = (3t, 3t^2, t^3 + 3, 5 - t, 7)$  or  $(3t, 3t^2, t^3 - 3, -5 - t, 7)$ , for some integer  $t$ , or
- $x = \frac{a^3 - 27c}{27b - 9a^2}$ ,

or we have

$$(10) \quad |x| \leq \max \left\{ 8 |b - a^2/3|^3 + |a/3|, |4a^3/27 + 2c - 2ab/3|^{3/4} + |a/3| \right\}.$$

Finally, as an analog of Theorem 1, we have

**Theorem 6.** *Let  $D$  be an odd integer. Then the number of solutions to the equation*

$$(11) \quad x^3 + D = 2^n$$

*in pairs of integers  $(x, n)$  is at most three.*

This last result is sharp as equation (11) with  $D = 3$  has precisely three solutions  $(x, n) = (-1, 1), (1, 2)$  and  $(5, 7)$ . In fairness, we should point out that this is not an analog of comparable generality to Theorem 1, in that the latter provides an upper bound for the number of solutions to the equation  $f(x) = 2^n$ , for all monic quadratic polynomials  $f$ , while the same is not true of Theorem 6 for monic cubics.

**3. Padé approximants to  $(1 - x)^\nu$ .** All our results in this paper have, at their heart, Padé approximation to the algebraic function  $(1 - x)^\nu$ , where  $\nu \in \mathbb{Q}/\mathbb{Z}$ . Recall that an  $[n_1/n_2]$ -Padé approximant to a function  $f(x)$  is a rational function  $p(x)/q(x)$ , where the numerator and denominator are polynomials with, say, integer coefficients, of degrees  $n_1$  and  $n_2$ , respectively, such that  $f(x)$  and  $p(x)/q(x)$  have the same MacLaurin series expansion up to degree  $n_1 + n_2$ , i.e. such that  $p(x)/q(x)$  is a good approximation to  $f(x)$  in a neighbourhood of  $x = 0$ .

Since the function  $p(x)/q(x)$  is a rational function with rational coefficients, it takes rational values for rational choices of its argument. In this way, we will be able to take a single suitably good approximation to, in our case, a particular algebraic number, and generate an infinite sequence of “good” approximations to the same number. In order to obtain sharp estimates for the quality of the approximations that are generated with these functions, we will use representations coming from contour integrals, as well as explicit descriptions of the Padé approximants as polynomials. Define

$$I_{n_1, n_2}(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{(1-zx)^{n_2}(1-zx)^{\nu}}{z^{n_1+1}(1-z)^{n_2+1}} dz$$

where  $n_1$  and  $n_2$  are positive integers,  $\gamma$  is a closed, positively oriented contour enclosing  $z = 0$  and  $z = 1$ , and  $|x| < 1$ . A straightforward application of Cauchy’s theorem yields that

$$(12) \quad I_{n_1, n_2}(x) = P_{n_1, n_2}(x) - (1-x)^{\nu} Q_{n_1, n_2}(x)$$

where  $P_{n_1, n_2}(x)$  and  $Q_{n_1, n_2}(x)$  are polynomials with rational coefficients of degrees  $n_1$  and  $n_2$ , respectively. In fact, examining the relevant residues, it is possible to show that

$$(13) \quad P_{n_1, n_2}(x) = \sum_{k=0}^{n_1} \binom{n_2 + \nu}{k} \binom{n_1 + n_2 - k}{n_2} (-x)^k$$

and

$$(14) \quad Q_{n_1, n_2}(x) = \sum_{k=0}^{n_2} \binom{n_1 - \nu}{k} \binom{n_1 + n_2 - k}{n_1} (-x)^k.$$

In particular, if we choose  $\nu \in \{1/3, 2/3\}$ , we have that  $P_{n_1, n_2}(x), Q_{n_1, n_2}(x) \in \mathbb{Z}[1/3][x]$ .

Our goal in the following sections will be twofold. First, we will derive estimates for the size of  $|I_{n_1, n_2}(x)|$  and  $|P_{n_1, n_2}(x)|$  using contour integral representations. Secondly, we will find lower bounds for the size of the greatest common divisor of the coefficients involved in  $|P_{n_1, n_2}(x)|$  and  $|Q_{n_1, n_2}(x)|$ .

**4. Bounding our approximants.** Here and henceforth, given a positive integer  $n_1$ , let us set  $n_2 = 4n_1 - \delta$ , where  $\delta \in \{0, 1\}$ , and write

$\nu = r/3$  where  $r \in \{1, 2\}$ . Define

$P_{n_1, \delta} = P_{n_1, 4n_1 - \delta}(3/128)$ ,  $Q_{n_1, \delta} = Q_{n_1, 4n_1 - \delta}(3/128)$  and  $I_{n_1, \delta} = I_{n_1, 4n_1 - \delta}(3/128)$ , so that, from (12),

$$(15) \quad I_{n_1, \delta} = P_{n_1, \delta} - \left( \frac{5}{2^{7/3}} \right)^r Q_{n_1, \delta}.$$

We further define  $F(z)$  by

$$(16) \quad F(z) = \frac{(1 - 3z/128)^4}{z(1 - z)^4}.$$

It is easy to show, via calculus, that  $F(z)$  attains its minimum for  $z \in (0, 1)$  at the value

$$(17) \quad \tau = \frac{1}{6} \left( 631 - 5\sqrt{15865} \right) = 0.20304\dots,$$

where we have

$$F := F(\tau) = \frac{243 (75 - \sqrt{15865})^4}{34359738368 (631 - 5\sqrt{15865}) (125 - \sqrt{15865})^4} = 11.97804\dots$$

Our argument will require upper bounds upon  $|P_{n_1, \delta}|$ ,  $|Q_{n_1, \delta}|$  and  $|I_{n_1, \delta}|$ ; from (15) it suffices to bound one of the first two of these, together with the last.

**Lemma 7.** *We have*

$$|P_{n_1, \delta}| < 1.26 \cdot F^{n_1} \quad \text{and} \quad |I_{n_1, \delta}| < \frac{4}{\pi} (2^{35} 3^{-5} F)^{-n_1},$$

*Proof.* We begin by separating the integral defining  $I_{n_1, \delta}(3/128)$  into two pieces, one involving a closed contour containing only the pole at  $z = 0$ . Taking  $\tau$  as in (17), we may write

$$P_{n_1, \delta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(1 - 3z/128)^{4n_1 - \delta + r/3}}{z^{n_1 + 1} (1 - z)^{4n_1 - \delta + 1}} dz$$

where  $\Gamma$  is the closed, positively oriented contour with  $|z| = \tau$ . Using the transformation  $z = \tau e^{i\theta}$ , we have that

$$|P_{n_1, \delta}| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{(1 - 3z/128)^{4n_1 - \delta + r/3}}{z^{n_1 + 1} (1 - z)^{4n_1 - \delta + 1}} \right| \tau d\theta$$

and so

$$|P_{n_1, \delta}| \leq \frac{1}{\tau^{n_1}} \max_{0 \leq \theta \leq 2\pi} \left| \frac{(1 - 3\tau e^{i\theta}/128)^{4n_1 - \delta + r/3}}{(1 - \tau e^{i\theta})^{4n_1 - \delta + 1}} \right|,$$

whereby

$$|P_{n_1, \delta}| \leq \frac{(1 - 3\tau/128)^{r/3}}{\tau^{n_1}(1 - \tau)} \left( \frac{1 - 3\tau/128}{1 - \tau} \right)^{4n_1 - \delta}.$$

Since  $r \in \{1, 2\}$ , the desired bound for  $|P_{n_1, \delta}|$  follows.

To bound  $|I_{n_1, \delta}|$ , we argue as in [1] to arrive at the identity

$$(18) \quad |I_{n_1, \delta}| = \frac{(3/128)^{5n_1 - \delta + 1}}{\pi} \int_0^1 \frac{v^{4n_1 - \delta + r/3} (1 - v)^{n_1 - r/3} dv}{(1 - 3(1 - v)/128)^{4n_1 - \delta + 1}}.$$

We may thus rewrite the integrand here as

$$f_{\delta, r}(v) \left( \frac{v^4(1 - v)}{(1 - 3(1 - v)/128)^4} \right)^{n_1 - 1}$$

where

$$f_{\delta, r}(v) = \frac{v^{4+r/3-\delta} (1 - v)^{1-r/3}}{(1 - 3(1 - v)/128)^{5-\delta}},$$

$\delta \in \{0, 1\}$  and  $r \in \{1, 2\}$ . Changing variable via  $v = 1 - z$ , we thus have

$$|I_{n_1, \delta}| \leq \frac{(3/128)^{5n_1 + 1 - \delta}}{\pi} \max\{f_{\delta, r}(v) : v \in (0, 1)\} F^{1 - n_1}.$$

Since a little calculus reveals that, in all cases,  $\max\{f_{\delta, r}(v) : v \in (0, 1)\} < 1/3$  and since we have  $F < 12$ , it follows that

$$|I_{n_1, \delta}| < \frac{4}{\pi} (2^{35} 3^{-5} F)^{-n_1},$$

as desired.  $\square$

### 5. Proof of a (restricted) irrationality measure : Theorem

**3.** Let us suppose that  $r \in \{1, 2\}$ ,  $s \in \{1, 3\}$ ,  $p$  and  $k$  are integers, and write  $m = 3k + r$ . From (13), (14) and the fact that

$$\binom{n \pm r/3}{j} 3^{\lfloor 3j/2 \rfloor} \in \mathbb{Z}$$



for all positive integers  $n$  and  $j$ , it follows that

$$3^{\lfloor n_1/2 \rfloor} 2^{7n_1} P_{n_1, \delta} \quad \text{and} \quad 3^{2n_1 - \delta} 2^{7(4n_1 - \delta)} Q_{n_1, \delta}$$

are both integers. Here,  $n_1$  is a positive integer to be chosen later. We set

$$\Pi_{n_1, \delta, r} = \gcd\{3^{\lfloor n_1/2 \rfloor} 2^{7n_1} P_{n_1, \delta}, 3^{2n_1 - \delta} 2^{28n_1 - 7\delta} Q_{n_1, \delta}\}$$

so that

$$A = \frac{3^{\lfloor n_1/2 \rfloor} 2^{7n_1}}{\Pi_{n_1, \delta, r}} P_{n_1, \delta} \quad \text{and} \quad B = \frac{3^{2n_1 - \delta} 2^{28n_1 - 7\delta}}{\Pi_{n_1, \delta, r}} Q_{n_1, \delta}$$

are integers. Equation (12) therefore implies that

$$\Pi_{n_1, \delta, r}^{-1} |I_{n_1, \delta}| = \left| \frac{A}{3^{\lfloor n_1/2 \rfloor} 2^{7n_1}} - \left( \frac{5}{27/3} \right)^r \frac{B}{3^{2n_1 - \delta} 2^{28n_1 - 7\delta}} \right|.$$

If we define

$$\Omega = \left| 1 - \frac{p}{s 2^{m/3}} \right|.$$

and

$$\Lambda = \left| \frac{p}{s 2^{m/3}} - \left( \frac{5}{27/3} \right)^r \frac{B}{A 2^{7(3n_1 - \delta)} 3^{2n_1 - \lfloor n_1/2 \rfloor - \delta}} \right|,$$

we have that

$$\Lambda < \Omega + \Pi_{n_1, \delta, r}^{-1} A^{-1} 3^{\lfloor n_1/2 \rfloor} 2^{7n_1} |I_{n_1, \delta}|$$

(note that the nonvanishing of  $A$  is a consequence of the contour integral representation for  $P_{n_1, n_2}(x)$ ). From Lemma 4 of Beukers [3], for one of our two choices of  $\delta \in \{0, 1\}$ , we have  $\Lambda \neq 0$ ; we fix  $\delta$  to be this value and choose

$$(19) \quad n_1 = 1 + \left\lfloor \frac{m - 7r + 21\delta}{63} \right\rfloor,$$

so that

$$7r/3 + 7(3n_1 - \delta) > \frac{m}{3}.$$

Since  $s \mid 3$ , we thus have

$$\Lambda \geq \left( A 2^{7(3n_1 - \delta + r/3)} 3^{2n_1 - \lfloor n_1/2 \rfloor - \delta} \right)^{-1}.$$

Combining our upper and lower bounds for  $\Lambda$ , we find that

$$(20) \quad 1 < \Omega\Lambda_1 + \Lambda_2,$$

where, upon substituting for  $A$ ,

$$\Lambda_1 = \Pi_{n_1, \delta, r}^{-1} 3^{2n_1 - \delta} 2^{7(4n_1 - \delta + r/3)} |P_{n_1, \delta}|$$

and

$$\Lambda_2 = \Pi_{n_1, \delta, r}^{-1} 3^{2n_1 - \delta} 2^{7(4n_1 - \delta + r/3)} |I_{n_1, \delta}|.$$

Applying Lemma 7, we thus have

$$\Lambda_1 < 1.26 \Pi_{n_1, \delta, r}^{-1} 2^{14/3} (3^2 2^{28} F)^{n_1}$$

and

$$\Lambda_2 < \frac{4}{\pi} \Pi_{n_1, \delta, r}^{-1} 2^{14/3} \left( \frac{3^7}{2^7 F} \right)^{n_1},$$

whereby, from (20),

$$(21) \quad \Omega = \left| 1 - \frac{p}{s 2^{m/3}} \right| > \frac{\Pi_{n_1, \delta, r} - \frac{4}{\pi} 2^{14/3} \left( \frac{3^7}{2^7 F} \right)^{n_1}}{1.26 2^{14/3} (3^2 2^{28} F)^{n_1}}.$$

In order for inequality (21) to be nontrivial, it remains therefore to show that

$$(22) \quad \liminf_{n_1 \rightarrow \infty} \frac{1}{n_1} \log \Pi_{n_1, \delta, r} > \log \left( \frac{3^7}{2^7 F} \right).$$

In the next section, we will in fact prove the following result.

**Proposition 8.** *For  $r \in \{1, 2\}$ ,  $n_1 \geq 497$  and  $\delta \in \{0, 1\}$ , we have that*

$$\Pi_{n_1, \delta, r} > 1.8^{n_1}.$$

Since we have that  $\frac{3^7}{2^7 F} < 1.427$ , inequality (22) follows as desired. Assuming Proposition 8 and further that  $k \geq 11000$ , so that  $m \geq 33000$ , (19) thus implies that  $n_1 \geq 497$ . We therefore have

$$\Pi_{n_1, \delta, r} - \frac{4}{\pi} 2^{14/3} \left( \frac{3^7}{2^7 F} \right)^{n_1} > 1.79^{n_1}$$

and hence, from (21),

$$\left| 1 - \frac{p}{s 2^{m/3}} \right| > 16166467234^{-n_1}.$$

From (19), we have that  $n_1 \leq (m + 77)/63$  and so

$$\left| 1 - \frac{p}{s 2^{m/3}} \right| > 16166467234^{-(m+77)/63} > 1.454^{-m},$$

where we have appealed to the fact that  $m \geq 33000$ . Since  $m = 3k + r$ , we thus have

$$\left| 2^{r/3} - \frac{p}{s 2^k} \right| > 2^{r/3} 1.454^{-3k-r} > 2^{-1.62k},$$

for both  $r \in \{1, 2\}$ ,  $k, p \in \mathbb{Z}$  and  $s \in \{1, 3\}$ .

For  $k \leq 100$ , a brute force search shows that the only approximations in this range that fail to satisfy the desired bound (7) have  $(r, s, p, k)$  in the following list (where we assume that  $\gcd(p, s2^k) = 1$  to avoid redundancy) :

$$(1, 1, 1, 0), (1, 1, 2, 0), (1, 1, 3, 1), (1, 1, 5, 2), (1, 3, 4, 0), (1, 3, 7, 1), (1, 3, 31, 3), \\ (1, 3, 61, 4), (1, 3, 121, 5), (2, 1, 1, 0), (2, 1, 2, 0), (2, 1, 3, 1), (2, 1, 3251, 11), \\ (2, 3, 5, 0), (2, 3, 19, 2), (2, 3, 305, 6).$$

In particular, in every case we have  $k < 12$ .

All that remains is to verify the inequality for, say,  $100 < k < 11000$ . We do this by considering the binary expansions of  $2^{r/3}$  and  $3 \cdot 2^{r/3}$ , for  $r \in \{1, 2\}$ , and searching for either unusually long strings of zeros or unusually long strings of ones (each of which would correspond to a very good approximation to  $2^{r/3}$  or  $3 \cdot 2^{r/3}$  by a rational with denominator a power of two). Such an argument is described in detail in [1] (see, in particular, Lemma 9.1 and the remarks following it). The fact that no such strings occur completes the proof of Theorem 3 (again, assuming Proposition 8).

**6. Arithmetic properties of the coefficients.** We now turn our attention to proving Proposition 8. To do this, we require first a good understanding of the content of the polynomials  $P_{n_1, n_2}(x)$  and  $Q_{n_1, n_2}(x)$ .

**Lemma 9.** *Let  $n_1$  be a positive integer,  $n_2 = 4n_1 - \delta$  for  $\delta \in \{0, 1\}$  and  $r \in \{1, 2\}$ . Suppose that  $p$  is prime, with*

$$(23) \quad p > \max\{\sqrt{3n_2 + 2}, 5\}.$$

*Assume that*

$$(24) \quad \left\{ \frac{n_1}{p} \right\} \in \left( \frac{2}{3}, \frac{3}{4} \right) \cup \left( \frac{5}{6}, 1 \right),$$

*if  $p \equiv r \pmod{3}$ , or*

$$(25) \quad \left\{ \frac{n_1}{p} \right\} \in \left( \frac{5}{12}, \frac{1}{2} \right) \cup \left( \frac{2}{3}, \frac{3}{4} \right) \cup \left( \frac{11}{12}, 1 \right),$$

*if  $p \equiv -r \pmod{3}$ . Then we have*

$$\text{ord}_p \binom{n_2 + r/3}{k} \binom{n_1 + n_2 - k}{n_2} \geq 1 \text{ for } 0 \leq k \leq n_1$$

*and*

$$\text{ord}_p \binom{n_1 - r/3}{k} \binom{n_1 + n_2 - k}{n_1} \geq 1 \text{ for } 0 \leq k \leq n_2.$$

*Proof.* We begin by considering the case when  $p \equiv r \pmod{3}$ . Observe that, from (24), we have that either  $\left\{ \frac{n_1}{p} \right\} \in \left[ \frac{2}{3} + \frac{r}{3p}, \frac{3}{4} \right)$ , whereby

$$(26) \quad \left\{ \frac{n_2}{p} \right\} = \left\{ \frac{4n_1 - \delta}{p} \right\} \in \left[ \frac{2}{3} + \frac{4r - 3}{3p}, 1 \right),$$

or that  $\left\{ \frac{n_1}{p} \right\} \in \left[ \frac{5}{6} + \frac{4r - 3}{6p}, 1 \right)$ , whence

$$(27) \quad \left\{ \frac{n_2}{p} \right\} = \left\{ \frac{4n_1 - \delta}{p} \right\} \in \left[ \frac{1}{3} + \frac{8r - 9}{3p}, 1 \right).$$

We conclude, in either case, that

$$(28) \quad \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} \geq \frac{7}{6} + \frac{20r - 21}{6p} \geq \frac{7}{6} - \frac{1}{6p} \geq 1 + \frac{1}{p},$$

where the last inequality follows from the assumption that  $p \geq 7$ . We proceed to show that  $p$  satisfying the hypotheses of the lemma have

positive valuation for the desired binomial coefficients. If  $k = 0$ , then

$$\binom{n_2 + r/3}{k} \binom{n_1 + n_2 - k}{n_2} = \binom{n_1 - r/3}{k} \binom{n_1 + n_2 - k}{n_1} = \binom{n_1 + n_2}{n_1}$$

and, since  $n_1 < n_2$ , our assumption that  $p^2 > 3n_2 + 2 > 3n_2$  implies that

$$\text{ord}_p \binom{n_1 + n_2}{n_1} = \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} - \left\{ \frac{n_1 + n_2}{p} \right\}.$$

It follows that  $\text{ord}_p \binom{n_1 + n_2}{n_1} \geq 1$  if and only if  $\left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} \geq 1$ , whereby, from (28), we conclude as desired.

Similarly, if  $k = 1$ , then the fact that

$$\text{ord}_p \binom{n_2 + r/3}{k} \binom{n_1 + n_2 - k}{n_2} \geq 1$$

is a consequence of the inequalities

$$\left\{ \frac{n_1 - 1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} \geq \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} - \frac{1}{p} \geq 1,$$

while

$$\text{ord}_p \binom{n_1 - r/3}{k} \binom{n_1 + n_2 - k}{n_1} \geq 1$$

follows from

$$\left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2 - 1}{p} \right\} \geq \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} - \frac{1}{p} \geq 1.$$

Let us next suppose that  $k \geq 2$ . From Lemma 4.5 of Chudnovsky [4], if  $n \in \mathbb{N}$  and  $p^2 > 3n + r$ , we have

$$\text{ord}_p \binom{n + r/3}{k} = \left[ \frac{n - q}{p} \right] - \left[ \frac{n - k - q}{p} \right] - \left[ \frac{k}{p} \right]$$

where  $q = (p - r)/3$ . It follows that

$$(29) \quad \text{ord}_p \binom{n_2 + r/3}{k} = \left\{ \frac{n_2 - q - k}{p} \right\} + \left\{ \frac{k}{p} \right\} - \left\{ \frac{n_2 - q}{p} \right\}$$

and

$$(30) \quad \text{ord}_p \binom{n_1 + n_2 - k}{n_2} = \left\{ \frac{n_2}{p} \right\} + \left\{ \frac{n_1 - k}{p} \right\} - \left\{ \frac{n_1 + n_2 - k}{p} \right\}.$$

Let us suppose that  $\text{ord}_p \binom{n_2+r/3}{k} \binom{n_1+n_2-k}{n_2} = 0$  and seek to derive a contradiction. From (29), we have  $\left\{ \frac{n_2-q}{p} \right\} \geq \left\{ \frac{k}{p} \right\}$  which, with (26) and (27), implies that

$$(31) \quad \left\{ \frac{n_2-q}{p} \right\} = \left\{ \frac{n_2}{p} \right\} - \frac{q}{p} = \left\{ \frac{n_2}{p} \right\} - \frac{p-r}{3p} \geq \left\{ \frac{k}{p} \right\}.$$

Similarly, from (30), we have

$$(32) \quad \left\{ \frac{n_2}{p} \right\} + \left\{ \frac{n_1-k}{p} \right\} < 1, \quad \text{i.e.} \quad \left\{ \frac{n_2}{p} \right\} + \left\{ \frac{n_1-k}{p} \right\} \leq 1 - \frac{1}{p}.$$

If  $\left\{ \frac{n_1}{p} \right\} < \left\{ \frac{k}{p} \right\}$ , since  $\left\{ \frac{n_1}{p} \right\} \geq \frac{2}{3} + \frac{1}{3p}$ , we have from (31), that

$$\left\{ \frac{n_2}{p} \right\} > \frac{2}{3} + \frac{1}{3p} + \frac{p-r}{3p} = 1 + \frac{1-r}{3p} \geq 1 - \frac{1}{3p},$$

an immediate contradiction. It follows that  $\left\{ \frac{n_1}{p} \right\} \geq \left\{ \frac{k}{p} \right\}$  and hence

$$\left\{ \frac{n_1-k}{p} \right\} = \left\{ \frac{n_1}{p} \right\} - \left\{ \frac{k}{p} \right\}.$$

We therefore have from (31) and (32) that

$$\left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} \leq 1 - \frac{1}{p} + \left\{ \frac{k}{p} \right\} \leq 1 - \frac{1}{p} + \left\{ \frac{n_2}{p} \right\} - \frac{p-r}{3p},$$

whence

$$\left\{ \frac{n_1}{p} \right\} \leq \frac{2p-3+r}{3p} < \frac{2}{3},$$

contradicting (24).

Similarly, we may write

$$(33) \quad \text{ord}_p \binom{n_1-r/3}{k} = \left\{ \frac{n_1-q-k}{p} \right\} + \left\{ \frac{k}{p} \right\} - \left\{ \frac{n_1-q}{p} \right\}$$

and

$$(34) \quad \text{ord}_p \binom{n_1+n_2-k}{n_1} = \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2-k}{p} \right\} - \left\{ \frac{n_1+n_2-k}{p} \right\}$$

where now  $q = (2p + r)/3$ . Let us suppose that

$$(35) \quad \text{ord}_p \binom{n_1 - r/3}{k} \binom{n_1 + n_2 - k}{n_1} = 0$$

so that, in particular, we have

$$(36) \quad \left\{ \frac{n_1 - q}{p} \right\} = \left\{ \frac{n_1}{p} \right\} - \frac{q}{p} = \left\{ \frac{n_1}{p} \right\} - \frac{2p + r}{3p} \geq \left\{ \frac{k}{p} \right\}.$$

It follows that we necessarily have

$$\left\{ \frac{k}{p} \right\} < 1 - \frac{2p + r}{3p} = \frac{1}{3} - \frac{r}{3p} \leq \left\{ \frac{n_2}{p} \right\},$$

whereby

$$\left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2 - k}{p} \right\} = \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} - \left\{ \frac{k}{p} \right\} \leq 1 - \frac{1}{p}.$$

Arguing as before, we find that

$$\left\{ \frac{n_2}{p} \right\} < 1 - \frac{1}{p} - \frac{2p + r}{3p} < \frac{1}{3} - \frac{1}{p},$$

again contradicting (26) and (27).

The argument for  $p \equiv -r \pmod{3}$  is essentially similar. Relation (25) implies that we have one of

$$(37) \quad \left\{ \frac{n_1}{p} \right\} \in \left[ \frac{5}{12} + \frac{9 - 4r}{12p}, \frac{1}{2} \right) \Rightarrow \left\{ \frac{n_2}{p} \right\} = \left\{ \frac{4n_1 - \delta}{p} \right\} \in \left[ \frac{2}{3} + \frac{6 - 4r}{3p}, 1 \right),$$

$$(38) \quad \left\{ \frac{n_1}{p} \right\} \in \left[ \frac{2}{3} + \frac{3 - r}{3p}, \frac{3}{4} \right) \Rightarrow \left\{ \frac{n_2}{p} \right\} = \left\{ \frac{4n_1 - \delta}{p} \right\} \in \left[ \frac{2}{3} + \frac{9 - 4r}{3p}, 1 \right)$$

or

$$(39) \quad \left\{ \frac{n_1}{p} \right\} \in \left[ \frac{11}{12} + \frac{9 - 4r}{12p}, 1 \right) \Rightarrow \left\{ \frac{n_2}{p} \right\} = \left\{ \frac{4n_1 - \delta}{p} \right\} \in \left[ \frac{2}{3} + \frac{6 - 4r}{3p}, 1 \right).$$

In each case, we may thus conclude that

$$(40) \quad \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} \geq \frac{13}{12} + \frac{33 - 20r}{12p} \geq \frac{13}{12} - \frac{7}{12p} \geq 1 + \frac{1}{p}.$$

Arguing as previously, (40) implies the desired conclusion almost immediately in case  $k = 0$  or  $1$ . Let us assume that  $k \geq 2$ . From Lemma 4.5 of Chudnovsky [4], we once again have (29) and (33), only this time with  $q = (2p - r)/3$  and  $q = (p + r)/3$ , respectively. If we suppose that  $\text{ord}_p\left(\binom{n_2+r/3}{k}\right) = 0$ , then

$$\left\{ \frac{n_2}{p} - \frac{2}{3} + \frac{r}{3p} \right\} \geq \left\{ \frac{k}{p} \right\},$$

whereby, from (37), (38) and (39),

$$(41) \quad \left\{ \frac{n_2}{p} \right\} \geq \frac{2}{3} - \frac{r}{3p} + \left\{ \frac{k}{p} \right\},$$

whence  $\left\{ \frac{k}{p} \right\} < 1/3 < \left\{ \frac{n_1}{p} \right\}$ . If also  $\text{ord}_p\left(\binom{n_1+n_2-k}{n_2}\right) = 0$ , we again have (32), and so, from (41),

$$\frac{2}{3} - \frac{r}{3p} + \left\{ \frac{n_1}{p} \right\} \leq \left\{ \frac{n_2}{p} \right\} + \left\{ \frac{n_1}{p} \right\} - \left\{ \frac{k}{p} \right\} \leq 1 - \frac{1}{p},$$

a contradiction. If, on the other hand, we assume (35), then both

$$\left\{ \frac{n_1 - \left(\frac{p+r}{3}\right)}{p} \right\} \geq \left\{ \frac{k}{p} \right\} \quad \text{and} \quad \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2 - k}{p} \right\} \leq 1 - \frac{1}{p}.$$

The first of these implies that

$$1 - \frac{1}{p} \geq \left\{ \frac{n_1}{p} \right\} \geq \frac{1}{3} + \frac{r}{3p} + \left\{ \frac{k}{p} \right\},$$

whence

$$\left\{ \frac{k}{p} \right\} \leq \frac{2}{3} - \frac{r+3}{3p}$$

and so, from (37), (38) and (39),

$$1 - \frac{1}{p} \geq \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2 - k}{p} \right\} = \left\{ \frac{n_1}{p} \right\} + \left\{ \frac{n_2}{p} \right\} - \left\{ \frac{k}{p} \right\} \geq \frac{1}{3} + \frac{r}{3p} + \left\{ \frac{n_2}{p} \right\},$$

whereby

$$\left\{ \frac{n_2}{p} \right\} \leq \frac{2}{3} - \frac{r+3}{3p}.$$

The resulting contradiction (to (37), (38) and (39)) completes the proof.  $\square$



To apply this result, we observe that if  $1 \leq c < d$  are integers, then we have  $\{n_1/p\} > c/d$  precisely when

$$p \in \bigcup_{k=0}^{\infty} \left( \frac{n_1}{k+1}, \frac{n_1}{k+c/d} \right).$$

Let us define

$$\theta(x, q, r) = \sum_{\substack{p \leq x \\ p \equiv r \pmod{q}}} \log p,$$

where the sum is over primes  $p$ . Fixing  $r \in \{1, 2\}$ , it follows from Lemma 9 that  $\log \Pi_{n_1, \delta, r}$  is bounded below by

$$(42) \quad L_{r, n_1} = \sum_{k=0}^{k_0} (T_{1,k} + T_{2,k} + T_{3,k} + T_{4,k} + T_{5,k}),$$

where

$$\begin{aligned} T_{1,k} &= \theta\left(\frac{n_1}{k+2/3}, 3, r\right) - \theta\left(\frac{n_1}{k+3/4}, 3, r\right), \\ T_{2,k} &= \theta\left(\frac{n_1}{k+5/6}, 3, r\right) - \theta\left(\frac{n_1}{k+1}, 3, r\right), \\ T_{3,k} &= \theta\left(\frac{n_1}{k+5/12}, 3, -r\right) - \theta\left(\frac{n_1}{k+1/2}, 3, -r\right), \\ T_{4,k} &= \theta\left(\frac{n_1}{k+2/3}, 3, -r\right) - \theta\left(\frac{n_1}{k+3/4}, 3, -r\right) \end{aligned}$$

and

$$T_{5,k} = \theta\left(\frac{n_1}{k+11/12}, 3, -r\right) - \theta\left(\frac{n_1}{k+1}, 3, -r\right),$$

for  $k_0 = \lceil \sqrt{\frac{n_1}{12}} \rceil - 2$  (we can actually, in most cases, use a slightly larger value for  $k_0$ ; it is simply chosen so that inequality (23) is satisfied). To estimate  $L_{r, n_1}$ , for large values of  $n_1$ , we will appeal to recent bounds on  $\theta(x, 3, r)$  due to the second author, Martin, O'Bryant and Rechnitzer [2]. In particular we use that

$$(43) \quad \left| \theta(x, 3, r) - \frac{x}{2} \right| < \begin{cases} 1.798158\sqrt{x} & \text{if } x \leq 10^{13} \\ 0.00144 \frac{x}{\log x} & \text{if } x > 10^{13}. \end{cases}$$

Note that together these inequalities imply that we have

$$\left| \theta(x, 3, r) - \frac{x}{2} \right| < 0.00144 \frac{x}{\log x}$$

for all  $x > 641239201$ . We will show that  $L_{r, n_1} > 0.58779 n_1$ , which immediately implies Proposition 8.

Let us assume first that  $n_1 > 2 \cdot 10^9$ . Then

$$T_{1,0} > \frac{n_1}{12} - 0.00144 \left( \frac{3n_1}{2 \log(3n_1/2)} + \frac{4n_1}{3 \log(4n_1/3)} \right) > 0.083145 n_1,$$

$$T_{2,0} > \frac{n_1}{10} - 0.00144 \left( \frac{6n_1}{5 \log(6n_1/5)} + \frac{n_1}{\log(n_1)} \right) > 0.099852 n_1,$$

$$T_{3,0} > \frac{n_1}{5} - 0.00144 \left( \frac{12n_1}{5 \log(12n_1/5)} + \frac{2n_1}{\log(2n_1)} \right) > 0.199714 n_1,$$

$$T_{4,0} > \frac{n_1}{12} - 0.00144 \left( \frac{3n_1}{2 \log(3n_1/2)} + \frac{4n_1}{3 \log(4n_1/3)} \right) > 0.083145 n_1,$$

$$T_{5,0} > \frac{n_1}{11} - 0.00144 \left( \frac{12n_1}{11 \log(12n_1/11)} + \frac{n_1}{\log(n_1)} \right) > 0.090768 n_1,$$

$$T_{1,1} > \frac{n_1}{70} - 0.00144 \left( \frac{3n_1}{5 \log(3n_1/5)} + \frac{4n_1}{7 \log(4n_1/7)} \right) > 0.014204 n_1,$$

and

$$T_{3,1} > \frac{n_1}{51} - 0.00144 \left( \frac{12n_1}{17 \log(12n_1/17)} + \frac{2n_1}{3 \log(2n_1/3)} \right) > 0.019513 n_1,$$

whereby  $L_{r, n_1} > 0.58779 n_1$  as desired. Next suppose that  $n_1 \leq 2 \cdot 10^9$ . Then, for each  $k \geq 0$ , we have

$$T_{1,k} > \frac{n_1}{2(3k+2)(4k+3)} - 1.798158 \left( \sqrt{\frac{n_1}{k+2/3}} + \sqrt{\frac{n_1}{k+3/4}} \right),$$

$$T_{2,k} > \frac{n_1}{2(6k+5)(k+1)} - 1.798158 \left( \sqrt{\frac{n_1}{k+5/6}} + \sqrt{\frac{n_1}{k+1}} \right),$$

$$T_{3,k} > \frac{n_1}{(2k+1)(12k+5)} - 1.798158 \left( \sqrt{\frac{n_1}{k+5/12}} + \sqrt{\frac{n_1}{k+1/2}} \right),$$

$$T_{4,k} > \frac{n_1}{2(3k+2)(4k+3)} - 1.798158 \left( \sqrt{\frac{n_1}{k+2/3}} + \sqrt{\frac{n_1}{k+3/4}} \right)$$

and

$$T_{5,k} > \frac{n_1}{2(12k+11)(k+1)} - 1.798158 \left( \sqrt{\frac{n_1}{k+11/12}} + \sqrt{\frac{n_1}{k+1}} \right).$$

If we suppose that  $10^6 \leq n_1 \leq 2 \cdot 10^9$ , then it is readily checked that the inequalities here are nontrivial (i.e. that the right-hand-sides are positive) for, in each case,  $0 \leq k \leq 4$ , whereby we find that, once again,

$$L_{r,n_1} \geq \sum_{k=0}^4 (T_{1,k} + T_{2,k} + T_{3,k} + T_{4,k} + T_{5,k}) > 0.58779 n_1.$$

It remains to treat values of  $n_1 < 10^6$ . We note that if we have  $a = 1$ ,  $n_1 = 496$  and  $n_2 = 4n_1 - 1$ , then

$$\Pi_{n_1,\delta,r}^{1/n_1} = 1.79954218\dots$$

and hence we cannot expect to extend Proposition 8 to smaller values of  $n_1$ . By direct (if slow) computation of  $\Pi_{n_1,\delta,r}$ , we find that the inequality of Proposition 8 is satisfied for each  $r \in \{1,2\}$ ,  $497 \leq n_1 \leq 1000$  and  $n_2 = 4n_1 - \delta$  with  $\delta \in \{0,1\}$ . For larger values of  $n_1$ , instead of relying upon the definition of  $\Pi_{n_1,\delta,r}$ , we appeal to the bound  $\log \Pi_{n_1,\delta,r} \geq L_{r,n_1}$ , where  $L_{r,n_1}$  is as defined in (42). For  $1000 < n_1 \leq 10000$  and  $r \in \{1,2\}$ , we check that, in each case,  $\exp(L_{r,n_1}/n_1) > 1.8$ . This takes roughly 20 minutes in Maple on an elderly Macbook Air. We find that the largest value of  $n_1$  in this range for which we have  $\exp(L_{r,n_1}/n_1) < 1.9$  corresponds to

$$\exp(L_{2,3319}/3319) = 1.89773\dots$$

This is unsurprising since, from (43), we have that  $L := \lim_{n_1 \rightarrow \infty} L_{r,n_1}/n_1$  is equal to

$$\sum_{k=0}^{\infty} \left( \frac{1}{(3k+2)(4k+3)} + \frac{1}{2(6k+5)(k+1)} + \frac{1}{(2k+1)(12k+5)} + \frac{1}{2(12k+11)(k+1)} \right).$$

Defining

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

we have that

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+x} \right), \quad \text{for } x \neq 0, -1, -2, \dots,$$

whence

$$L = \psi(1) + \psi(3/4) + \psi(1/2)/2 - \psi(2/3) - \psi(5/6)/2 - \psi(5/12)/2 - \psi(11/12)/2.$$

Using known identities for  $\psi$ , we thus have

$$L = -\gamma - 3 \ln(2) + \frac{\pi}{2} - \frac{5\pi\sqrt{3}}{12} + \frac{9 \ln 3}{4} - \frac{\psi(5/12)}{2} - \frac{\psi(11/12)}{2} = 0.70264 \dots,$$

and so

$$\lim_{n_1 \rightarrow \infty} \exp(L_{r, n_1}/n_1) = 2.019084 \dots$$

To finish the computation verifying the inequality  $\exp(L_{r, n_1}/n_1) > 1.8$  for  $10000 < n_1 < 10^6$ , we employ the “bootstrapping” procedure described in detail in Section 7 of [1], which exploits that  $\lim_{n_1 \rightarrow \infty} \exp(L_{r, n_1}/n_1)$  greatly exceeds 1.8 – by way of example

$$\exp(L_{1, 10000}/10000) > 2.006523$$

– together with the fact that the difference between  $L_{r, n_1}$  and  $L_{r, n_1+k}$  is “small”, provided  $n_1$  is much larger than  $k$ . This enables us to significantly reduce the number of times we actually compute  $L_{r, n_1}$ . Full details are available from the authors on request. This completes the proof of Proposition 8..

**7. Proof of Corollary 4.** To go from Theorem 3 to Corollary 4 is straightforward. Suppose that  $x$  and  $n$  are integers with  $x^3 \neq 27 \cdot 2^n$  and  $x \notin \{4, 5, 8, 15, 19, 38, 121\}$ . We may suppose further that  $x$  is positive since our desired conclusion is trivial otherwise. If  $n \equiv 0 \pmod{3}$ , say  $n = 3n_0$ , we have

$$|x^3 - 27 \cdot 2^n| \geq x^3 - (x-1)^3 > 3x^3 - 3x^2 \geq 3^{5/3} \cdot x^{4/3},$$

where the last inequality holds for  $x \geq 3$  (and, for  $x = 1$  or  $2$ , the desired result follows immediately). We may thus suppose that  $n = 3n_0 + r$  for  $r \in \{1, 2\}$ , and so

$$|x^3 - 27 \cdot 2^n| = 3 \cdot 2^{n_0} \left| 2^{r/3} - \frac{x}{3 \cdot 2^{n_0}} \right| \cdot \left( x^2 + 3 \cdot x 2^{n_0+r/3} + 9 \cdot 2^{2n_0+2r/3} \right).$$

If  $|x - 3 \cdot 2^{n_0+r/3}| > 1$ , then, once again, we have

$$|x^3 - 27 \cdot 2^n| > 3x^3 - 3x^2 \geq 3^{5/3} \cdot x^{4/3}.$$

Otherwise, applying Theorem 3, we thus have

$$|x^3 - 27 \cdot 2^n| > 3 \cdot 2^{-0.62n_0} (x^2 + x(x-1) + x(x-1)^2),$$

at least provided  $n_0 > 12$ . Since

$$2^{0.62n_0} \leq \left(\frac{x+1}{3}\right)^{0.62},$$

we obtain inequality (8), after a little work. The values  $n_0 \leq 12$  correspond to  $n \leq 38$ . For these, we readily check that (8) is satisfied, except for  $x \in \{4, 5, 8, 15, 19, 38, 121\}$ .

**8. Proof of Theorem 5.** We next proceed with the proof of Theorem 5. Let us suppose that  $a, b$  and  $c$  are given integers and that there exist integers  $x$  and  $n$  such that

$$x^3 + ax^2 + bx + c = \pm 2^n.$$

Writing  $u = x + a/3$ , we find that

$$(44) \quad u^3 + (b - a^2/3)u + (2a^3/27 + c - ab/3) = (-1)^\delta 2^n,$$

where now either  $u$  or  $3u$  is an integer and  $\delta \in \{0, 1\}$ .

If  $u^3 = (-1)^\delta 2^n$  then, from (44),

$$(b - a^2/3)(x + a/3) = ab/3 - c - 2a^3/27$$

so that

$$(b - a^2/3)x = -c + a^3/27.$$

If  $b = a^2/3$  then we necessarily have that  $a = 3t$  for some integer  $t$ , whereby  $b = 3t^2$  and  $c = t^3$ , so that  $x^3 + ax^2 + bx + c = (x+t)^3$ . Otherwise, we conclude that

$$x = \frac{a^3 - 27c}{27b - 9a^2}.$$

Next, to treat the cases where  $3 \mid a$  and

$$x = u - a/3 \quad \text{for} \quad |u| \in \{4, 5, 8, 15, 19, 38, 121\},$$

let us suppose that for one of these choices of  $x$  we have

$$(45) \quad |x| > \max \left\{ 8 |b - a^2/3|^3 + |a/3|, |4a^3/27 + 2c - 2ab/3|^{3/4} + |a/3| \right\}.$$

Since  $|x| \leq |a/3| + |u|$  we thus have

$$(46) \quad \max \left\{ 8 |b - a^2/3|^3, |4a^3/27 + 2c - 2ab/3|^{3/4} \right\} < |u|.$$

It follows that  $b = a^2/3 + b_0$  for some integer  $b_0$  with  $|b_0| < \frac{1}{2} |u|^{1/3}$ . Since we also have

$$\left| c - \frac{ab_0}{3} - \frac{a^3}{27} \right| < \frac{1}{2} |u|^{4/3},$$

we may write

$$c = \frac{ab_0}{3} + \frac{a^3}{27} + c_0,$$

where  $c_0$  is an integer with  $|c_0| < \frac{1}{2} |u|^{4/3}$ . Writing  $a = 3t$ , for  $t$  an integer, we thus have

$$(u - t)^3 + 3t(u - t)^2 + (3t^2 + b_0)(u - t) + t^3 + tb_0 + c_0 = \pm 2^n,$$

whence

$$(47) \quad u^3 + b_0u + c_0 = \pm 2^n,$$

where

$$|u| \in \{4, 5, 8, 15, 19, 38, 121\}, \quad |b_0| < \frac{1}{2} |u|^{1/3} \quad \text{and} \quad |c_0| < \frac{1}{2} |u|^{4/3}.$$

A short computation reveals that (47) is satisfied only for

$$(u, b_0, c_0) \in \{(\pm 4, 0, 0), \pm(5, 0, 3), (\pm 8, 0, 0)\},$$

the first and last of which correspond to  $x^3 + ax^2 + bx + c = (x + t)^3$ . The case  $(u, b_0, c_0) = \pm(5, 0, 3)$  leads to  $(a, b, c, x) = (3t, 3t^2, t^3 + 3, 5 - t)$  and  $(3t, 3t^2, t^3 - 3, -5 - t)$ .

If we suppose finally that  $u^3 \neq (-1)^\delta 2^n$  and  $x \neq u - a/3$  for any  $u$  satisfying  $|u| \in \{4, 5, 8, 15, 19, 38, 121\}$ , from the fact that

$$|u^3 - (-1)^\delta 2^n| = |(b - a^2/3)u + (2a^3/27 + c - ab/3)| > 0$$

we thus have, in all cases, applying Corollary 4 to  $|3u|^3 - 27 \cdot 2^n|$ , that

$$u^{4/3} \leq |(b - a^2/3)u + (2a^3/27 + c - ab/3)| \leq 2 \max \{|b - a^2/3| |u|, |2a^3/27 + c - ab/3|\}$$

and so

$$|u| \leq \max \left\{ 8 |b - a^2/3|^3, |4a^3/27 + 2c - 2ab/3|^{3/4} \right\},$$

whereby

$$|x| \leq \max \left\{ 8 |b - a^2/3|^3 + |a/3|, |4a^3/27 + 2c - 2ab/3|^{3/4} + |a/3| \right\}.$$

This completes the proof of Theorem 5.

**9. Proof of Theorem 6.** In this section, we will prove Theorem 6. Let us begin by supposing that  $D$  is an odd integer and that we have

$$x_i^3 + D = 2^{k_i}, \quad i \in \{1, 2, 3, 4\},$$

with

$$x_1 < x_2 < x_3 < x_4 \quad \text{and} \quad k_1 < k_2 < k_3 < k_4.$$

Then, for each  $i \in \{1, 2, 3\}$ ,

$$(x_{i+1} - x_i)(x_{i+1}^2 + x_{i+1}x_i + x_i^2) = x_{i+1}^3 - x_i^3 = 2^{k_i}(2^{k_{i+1}-k_i} - 1),$$

whereby we may write

$$x_{i+1} = x_i + a_i \cdot 2^{k_i}$$

with  $a_i$  a positive integer. Substituting this into  $2^{k_{i+1}} = x_{i+1}^3 + D$ , we find that

$$(48) \quad 2^{k_{i+1}-k_i} = 1 + 3a_i x_i^2 + 3a_i^2 2^{k_i} x_i + a_i^3 2^{2k_i}.$$

Let us suppose first that  $D$  is positive and write  $x_i = -D^{1/3} + y_i$  for  $y_i$  a positive real number. From (48), we have

$$2^{k_{i+1}-k_i} = 1 + a_i \cdot \left( 3D^{2/3} - 3a_i 2^{k_i} D^{1/3} + a_i^2 2^{2k_i} \right) + 3a_i y_i \left( y_i + a_i 2^{k_i} - 2D^{1/3} \right).$$

Applying the arithmetic-geometric mean inequality to the first bracketed term on the right-hand-side of this equation, we thus have that

$$(49) \quad 2^{k_{i+1}-k_i} \geq 1 + \left( 2\sqrt{3} - 3 \right) a_i^2 D^{1/3} 2^{k_i} + 3a_i y_i \left( y_i + a_i 2^{k_i} - 2D^{1/3} \right).$$

Notice that

$$2^{k_i} = x_i^3 + D = 3y_i D^{2/3} - 3y_i^2 D^{1/3} + y_i^3$$

and hence

$$y_i + a_i 2^{k_i} - 2D^{1/3} = y_i + 3a_i y_i D^{2/3} - 3a_i y_i^2 D^{1/3} + a_i y_i^3 - 2D^{1/3}.$$

If we have  $y_1 < 1$  then, since

$$y_2 - y_1 = x_2 - x_1 = a_1 \cdot 2^{k_1} \geq 1,$$

we thus have  $y_2 > 1$  and hence, in all cases, may write  $y_2 = D^\theta$  for  $\theta > 0$ . Since, again by the arithmetic-geometric mean inequality, the function

$$f(\theta) = 3D^{2/3+\theta} - 3D^{1/3+2\theta} + D^{3\theta}$$

is monotone increasing as a function of  $\theta$ , we thus have

$$2^{k_2} > 3D^{2/3} - 3D^{1/3} + 1 > 2D^{2/3},$$

at least assuming that  $D \geq 27$ . From (49), it follows that

$$2^{k_3} > 2D^{2/3} \left( (2\sqrt{3} - 3) 2D + 3(2D^{2/3} - 2D^{1/3}) \right) > 4(2\sqrt{3} - 3) D^{5/3} > 1.8 \cdot D^{5/3}.$$

We therefore have  $x_3 > D^{5/9}$  and so, again applying (49), we find that  $2^{k_4} > 1.5 \cdot D^{11/3}$  and so

$$x_4^3 > 1.5 \cdot D^{11/3} - D > D^{11/3},$$

since we may assume that  $D \geq 2$ . We thus have, from Corollary 4 (where we take  $x = 3x_4$ ), that

$$D = |x_4^3 - 2^{k_4}| \geq x_4^{4/3} > (D^{11/3})^{4/9} = D^{44/27},$$

a contradiction.

Suppose next that  $D < 0$  (so that  $x_2 > x_1 > |D|^{1/3}$ ). From (48), we have that  $2^{k_2} > 2|D|^{2/3}$  and hence

$$2^{k_3} > 2^{k_2} (3x_2^2 + 3 \cdot 2^{k_2} x_2 + 2^{2k_2}) > 8|D|^2.$$

A final appeal to (48) implies that

$$2^{k_4} > 2^{k_3} (3x_3^2 + 3 \cdot 2^{k_3} x_3 + 2^{2k_3}) > 512|D|^6.$$



We thus have, again appealing to Corollary 4, that

$$|D| = |x_4^3 - 2^{k_4}| \geq x_4^{4/3} = (2^{k_4} + |D|)^{4/9} > (512|D|^6)^{4/9} > |D|^{8/3},$$

a contradiction. This completes the proof of Theorem 6.

It is perhaps worthwhile noting that we know of only three odd values of  $D$  for which the equation  $x^3 + D = 2^n$  has even as many as two solutions in integers  $x$  and  $n$ , namely  $D = -215$  (with  $(x, n) = (6, 0)$  and  $(7, 7)$ ),  $D = 1$  with  $(x, n) = (0, 0)$  and  $(1, 1)$ , and  $D = 3$ , with  $(x, n) = (-1, 1)$ ,  $(1, 2)$  and  $(5, 7)$ .

**10. Thue-Mahler equations.** As noted earlier, the equation

$$(50) \quad x^3 - xy^2 + 8y^3 = 2^k,$$

which generalizes (1), has itself at most finitely many solutions in integers  $(x, y, n)$ , which may be determined effectively following arguments of Tzanakis and de Weger [13], based upon lower bounds for linear forms in complex and  $p$ -adic logarithms, together with computational techniques from Diophantine approximation. Hambrook [6] has an implementation of such an approach which works well in the case of few primes and low degree forms (i.e. precisely the situation in which we find ourselves); appealing to his Thue-Mahler solver, the only coprime solutions to (50) are with  $(x, y)$  one of

$$\begin{aligned} &(-113, 53), (-19, 9), (-2, 1), (-1, 1), (-1, 5), (0, 1), (1, 0), \\ &(1, 1), (3, -1), (3, 1), (5, 1), (7, -3), (8, 1), (13, -6). \end{aligned}$$

Similarly, the equation

$$x^3 - 13xy^2 + 20y^3 = 2^k.$$

has corresponding solutions with  $(x, y)$  among

$$\begin{aligned} &(-21, 5), (-4, 1), (-3, 1), (-1, 1), (1, 0), (1, 1), (2, 1), \\ &(3, 1), (4, 1), (7, 3), (11, 5), (13, 1), (19, -3). \end{aligned}$$

There exist completely general bounds for the number of solutions to Thue-Mahler equations that depend only upon the degree of the given form  $F$  and the number of primes on the right hand side of the equation  $F(x, y) = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ . Along these lines, let us note that Evertse [5] has shown that if  $F$  is an irreducible cubic form and  $p$  is a fixed prime,

then the equation

$$|F(x, y)| = p^n$$

has at most  $7^{60} + 6 \cdot 7^4$  solutions in integers. This bound, while admirably uniform, exceeds  $10^{50}$ .

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