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Pillai's conjecture revisited

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Abstract

We prove a generalization of an old conjecture of Pillai (now a theorem of Stroeker and Tijdeman) to the effect that the Diophantine equation $3^x - 2^y = c$ has, for $|c| > 13$, at most one solution in positive integers x and y. In fact, we show that if N and c are positive integers with $N \geqslant 2$, then the equation $|(N+1)^x - N^y| = c$ has at most one solution in positive integers x and y, unless $(N, c) \in \{(2,1), (2,5), (2,7), (2,13), (2,23), (3,13)\}$. Our proof uses the hypergeometric method of Thue and Siegel and avoids application of lower bounds for linear forms in logarithms of algebraic numbers.

MSC: primary 11D61; 11D45

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1. Introduction

Let us suppose that a, b and c are fixed nonzero integers and consider the exponential Diophantine equation

$$a^x - b^y = c. \tag{1.1}$$

In 1936, Herschfeld [He] showed, if $(a, b) = (3, 2)$ and $|c|$ is sufficiently large, that Eq. (1.1) has at most a single solution in positive integers x and y. Later that year, Pillai [Pi2] (see also [Pi1]) extended this result to general (a, b) with $\gcd(a, b) = 1$ and $a > b \geqslant 2$, provided $|c| > c_0(a, b)$. Since Pillai's work (and, for that matter, Herschfeld's) depends upon Siegel's sharpening of Thue's theorem on rational

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approximation to algebraic numbers, it is ineffective (in that it is not possible, from the proof, to compute $c_0(a,b)$). In the special case where $a=3$ and $b=2$, Pillai [Pi3], conjectured that $c_0(3,2)=13$, noting the equations:

$$3-2=3^2-2^3=1,\ 3-2^3=3^3-2^5=-5 \text{ and } 3-2^4=3^5-2^8=-13.$$

This conjecture remained open until 1982 when Stroeker and Tijdeman [StTi] (see also de Weger [dW]) proved it using lower bounds for linear forms in logarithms of algebraic numbers, à la Baker (an earlier claim was made without proof by Chein [Ch]). Subsequently, Scott [Sc] gave an elementary proof, exploiting properties of integers in quadratic fields.

Our object in this paper is to prove a generalization of Pillai's conjecture which is not amenable to the techniques of [Sc], avoiding the use of linear forms in logarithms. In fact, we will utilize bounds for the fractional parts of powers of rational numbers, established via the hypergeometric method. Based upon techniques of Thue and Siegel, using rational function approximation to hypergeometric series, this approach was, at least in principle, available to Herschfeld and Pillai. We obtain

Theorem 1.1. *If N and c are positive integers with $N \geqslant 2$, then the equation*

$$|(N+1)^x - N^y| = c$$

has at most one solution in positive integers x and y, unless

$$(N,c) \in \{(2,1),(2,5),(2,7),(2,13),(2,23),(3,13)\}.$$

In the first two of these cases, there are precisely 3 solutions, while the last four cases have 2 solutions apiece.

These exceptional cases correspond to the equations:

$$3-2=3^2-2^3=2^2-3=1,$$

$$3^2-2^2=2^3-3=2^5-3^3=5,$$

$$3^2-2=2^4-3^2=7,$$

$$2^4-3=2^8-3^5=13,$$

$$3^3-2^2=2^5-3^2=23.$$

For a good exposition of these and related subjects on exponential Diophantine equations, the reader is directed to the books of Ribenboim [Ri] and Shorey and Tijdeman [ShTi].

2. Fractional parts of powers of rationals

If u is a real number, let us denote by $||u||$, the distance from u to the nearest integer; i.e.

$$||u|| = \min\{|u - M| : M \in \mathbb{Z}\}.$$

In 1981, Beukers [Beu] was the first to apply the hypergeometric method of Thue and Siegel to the problem of obtaining bounds for the fractional parts of powers of rational numbers. In particular, he deduced lower bounds for $||(\frac{N+1}{N})^k||$, for $k \in N$ and $N \geqslant 2$. We note that the case $N = 2$ has special importance for the quantity $g(k)$ in Waring's problem (see e.g. [HW, p. 337]). For our purposes, we will use a result of the author [Ben] which slightly refines the corresponding inequality of [Beu]:

Proposition 2.1. *If N and k are integers with $4 \leqslant N \leqslant k \cdot 3^k$, then*

$$\left|\left|\left(\frac{N+1}{N}\right)^k\right|\right| > 3^{-k}.$$

In case $N = 2$, by applying the techniques of [Ben,Beu] (as done in nonexplicit fashion in [Du]), in combination with some (nontrivial) computation, we may prove

Proposition 2.2. *If $k \geqslant 5$ is an integer, then*

$$\left|\left|\left(\frac{3}{2}\right)^k\right|\right| > 2^{-0.8k}.$$

A result of this flavour was obtained by Dubitskas [Du], with the exponent 0.8 replaced by $0.793\ldots$, for $k \geqslant k_0$, where the last constant is effectively computable. In our context, we use Propositions 2.1 and 2.2 to show, if

$$(N+1)^{x_2} - N^{y_2} = (N+1)^{x_1} - N^{y_1} \tag{2.1}$$

for x_1, x_2, y_1 and y_2 positive integers with, say, $x_2 > x_1$, that

$$|(N+1)^{x_2} - N^{y_2}| > (N/3)^{x_2} \tag{2.2}$$

if $N \geqslant 4$, and

$$|3^{x_2} - 2^{y_2}| > 2^{x_2/5}, \tag{2.3}$$

provided $(x_2, y_2) \neq (2, 3)$. To see these, first note that necessarily $x_2 < y_2$. If not, we would have

$$(N+1)^{x_1} - N^{y_1} = (N+1)^{x_2} - N^{y_2} \geqslant (N+1)^{x_2} - N^{x_2} \geqslant (N+1)^{x_1+1} - N^{x_1+1}$$

and so

$$(N+1)^{x_1} > (N+1)^{x_1+1} - N^{x_1+1}.$$

This implies that $(N+1)^{x_1} < N^{x_1}$, an immediate contradiction. Since

$$|(N+1)^{x_2} - N^{y_2}| = |(N+1)^{x_2} - N^{y_2-x_2}N^{x_2}| \geqslant N^{x_2} \left\| \left(\frac{N+1}{N}\right)^{x_2} \right\|,$$

we easily obtain (2.3) from Proposition 2.2. To derive (2.2), we consider the cases $N \leqslant x_2 3^{x_2}$ and $N > x_2 3^{x_2}$ separately. In the first instance, (2.2) is immediate. If, on the other hand, $N > x_2 3^{x_2}$, then, since

$$(N+1)^{x_2} - N^{y_2} < (N+1)^{x_2} - N^{x_2+1} = N^{x_2}\left(\left(\frac{N+1}{N}\right)^{x_2} - N\right),$$

we have

$$(N+1)^{x_2} - N^{y_2} < N^{x_2}\left(\left(\frac{N+1}{N}\right)^{N/3} - N\right).$$

Since this last quantity is negative, it follows that

$$|(N+1)^{x_2} - N^{y_2}| = N^{y_2} - (N+1)^{x_2} > N^{x_2}(N - e^{1/3}) > (N/3)^{x_2}$$

as desired.

We will apply inequalities (2.2) and (2.3) to show, if (x_1, y_1, x_2, y_2) is a solution to (2.1), then $x_2 - x_1$ and $y_2 - y_1$ are relatively small. In the next section, we will derive lower bounds upon these quantities, leading, in most cases, to a contradiction.

3. A gap principle

If (x_1, y_1, x_2, y_2) is a solution to (2.1), with $x_2 > x_1$, then we have both

$$(N+1)^{x_2-x_1} \equiv 1 \pmod{N^{y_1}} \quad \text{and} \quad N^{y_2-y_1} \equiv 1 \pmod{(N+1)^{x_1}}. \tag{3.1}$$

We will use these congruences to bound $x_2 - x_1$ and $y_2 - y_1$ from below, via the following lemma (where we write $v_p(m)$ for the p-adic valuation of m):

Lemma 3.1. *Let x, y and N be positive integers with $N \geqslant 2$. If*

$$N^y \equiv 1 \pmod{(N+1)^x}, \tag{3.2}$$

then y is divisible by

$$\begin{cases} 2(N+1)^{x-1} & \text{if } N \text{ is even or } x = 1, \\ 2^{2-x}(N+1)^{x-1} & \text{if } N \equiv 1 \pmod 4 \text{ and } x \leqslant v_2(N-1)+1, \\ 2^{1-v_2(N-1)}(N+1)^{x-1} & \text{otherwise.} \end{cases}$$

If, on the other hand, we have

$$(N+1)^x \equiv 1 \pmod{N^y}, \tag{3.3}$$

then x is divisible by

$$\begin{cases} N^{y-1} & \text{if } N \text{ is odd, } N \equiv 0 \pmod 4 \text{ or } y = 1, \\ 2^{2-y}N^{y-1} & \text{if } N \equiv 2 \pmod 4 \text{ and } y \leqslant v_2(N+2)+1, \\ 2^{1-v_2(N+2)}N^{y-1} & \text{otherwise.} \end{cases}$$

Proof. We are grateful to the anonymous referee for suggesting the proof of this lemma in its current form. Let us begin by supposing that x, y and N satisfy (3.2). It follows that y is even and, in fact, if p is a prime divisor of $N+1$ and $z \geqslant 1$ is any integer, we have

$$N^{2z} \equiv 1 \pmod p \text{ and } N^{2z} \equiv 1 \pmod 4 \text{ (if } p = 2).$$

The p-adic valuations of $\log_p(N^{2z})$ and $N^{2z}-1$ are thus equal. Since $\log_p(N^{2z}) = z\log_p(N^2)$, we obtain the desired result, provided N is even. If N is odd, the necessary conclusion is a consequence of the fact that

$$v_2(\log_2(N^2)) = v_2(N^2-1) = v_2(N+1) + v_2(N-1).$$

The analogous statement for x, y and N satisfying (3.3) follows from a similar argument upon noting that

$$v_2(\log_2((N+1)^2)) = v_2((N+1)^2-1) = v_2(N) + v_2(N+2). \qquad \square$$

4. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1. Let us first consider the equation

$$(N+1)^{x_2} - N^{y_2} = N^{y_1} - (N+1)^{x_1}.$$

If (x_1, y_1, x_2, y_2) is a solution to this, in positive integers, then N divides

$$(N+1)^{x_2} + (N+1)^{x_1}.$$

Since this latter quantity is congruent to 2 modulo N, it follows that $N = 2$. By Theorem II of Pillai [Pi3], the equation

$$3^{x_2} - 2^{y_2} = 2^{y_1} - 3^{x_1} = c$$

has only the solutions

$$(x_1, y_1, x_2, y_2, c) = (1,2,2,3,1), (1,2,1,1,1), (1,3,2,2,5),$$
$$(3,5,2,2,5), (2,4,2,1,7), (2,5,3,2,23)$$

(where, without loss of generality, we assume that $c > 0$). We may thus restrict attention to Eq. (2.1) (with, again, $x_2 > x_1$). We will combine Lemma 3.1 with inequalities (2.2) and (2.3) to finish the proof of Theorem 1.1.

Let us begin by supposing that $N = 2$ and

$$3^{x_2} - 2^{y_2} = 3^{x_1} - 2^{y_1} = c, \tag{4.1}$$

where $x_2 > x_1$. Considering this equation modulo 3, we find that $y_1 \equiv y_2 \pmod 2$. Let us suppose first that $y_1 = 1$. Modulo 8, (4.1) implies that x_2 is even and x_1 odd. If $x_1 = 1, 3^2 - 2^3 = 3 - 2 = 1$ and (2.3) implies that there are no additional solutions to $3^x - 2^y = 1$. Otherwise, from (2.3),

$$3^{x_1} - 2 = 3^{x_2} - 2^{y_2} > 2^{x_2/5}$$

and so, since $3^{x_2} - 2^{y_2} > 0$ implies that $x_2 > \frac{\log 2}{\log 3} y_2$, (2.3) and Lemma 3.1 give

$$3^{x_1} > 2^{\frac{\log 2}{5 \log 3}(2 \cdot 3^{x_1 - 1} + 1)} + 2$$

and so, since $x_1 > 1$ is odd, $x_1 = 3$. We thus have $3^{x_2} - 2^{y_2} \equiv 0 \pmod 5$, contradicting x_2 even and y_2 odd.

Next suppose that $y_1 \geqslant 2$. Considering Eq. (4.1) modulo 8 implies that $y_1 \geqslant 3$ and that $x_1 \equiv x_2 \pmod 2$. From (2.3) and Lemma 3.1, we have either

$$3^{x_1} > 2^{\frac{\log 2}{5 \log 3}(2 \cdot 3^{x_1 - 1} + y_1)} + 2^{y_1}$$

or

$$2^{y_1} > 2^{(2^{y_1 - 2} + x_1)/5} + 3^{x_1},$$

according to whether $c > 0$ or $c < 0$, respectively. In the first instance, since we have already treated the case $c = 1$, it is straightforward to show that

$$(x_1, y_1) \in \{(3,3), (3,4)\}$$

while, in the second, we necessarily have

$$(x_1, y_1) \in \{(1,3),(1,4),(2,4),(1,5),(2,5),(3,5),(1,6),(2,6),(3,6),(1,7),(2,7)\}.$$

It follows that, in every case, $|c| \leqslant 125$ and so, from (2.3), all solutions to $3^x - 2^y = c$, for the values of c under consideration, satisfy $y \leqslant 34$. A routine computation confirms that the only additional solutions obtained correspond to $3 - 2^3 = 3^3 - 2^5 = -5$ and $3 - 2^4 = 3^5 - 2^8 = -13$. This completes our treatment of Eq. (2.1) for $N = 2$ (and hence for $N = 3$ and 8, as well).

Let us next suppose that $N \geqslant 4$ and

$$(N+1)^{x_2} - N^{y_2} = (N+1)^{x_1} - N^{y_1} > 0.$$

From (2.2), we have

$$(N+1)^{x_1} = (N+1)^{x_2} - N^{y_2} + N^{y_1} > (N/3)^{x_2} + N^{y_1}, \tag{4.2}$$

and, since $(N+1)^{x_2} > N^{y_2}$, we may conclude that

$$(N+1)^{x_1} > (N/3)^{\frac{\log(N)}{\log(N+1)}y_2} + N^{y_1}. \tag{4.3}$$

Note that (4.2) implies $x_1 \geqslant 2$. If $N = 4$, Lemma 3.1 yields $y_2 - y_1 \geqslant 10$. If $x_1 = 2$ and $y_2 = 11$, then (4.3) gives $y_1 = 1$ and hence $5^{x_2} = 4^{11} + 21 = 4\,194\,325$, which is a tad unlikely. If $x_1 = 2$ and $y_2 \geqslant 12$, we have, from $5^{x_2} > 4^{y_2}$, that $x_2 \geqslant 11$, whereby (4.2) yields a contradiction. It follows that $x_1 \geqslant 3$ and so Lemma 3.1 gives $y_2 - y_1 \geqslant 2 \cdot 5^{x_1 - 1}$, whence, from (4.3),

$$5^{x_1} > (4/3)^{\frac{\log 4}{\log 5}}(5^{x_1-1} + y_1) + 4^{y_1}.$$

Since $y_1 \geqslant 1$, this is a contradiction. Similarly, if $N \geqslant 5$, we have $y_2 - y_1 \geqslant 6$. If $7 \leqslant y_2 \leqslant 9$, the inequality $(N+1)^{x_2} > N^{y_2}$ contradicts $x_2 < y_2$ (see the remarks following (2.3)). We therefore have $y_2 \geqslant 10$ and so, again from $(N+1)^{x_2} > N^{y_2}, x_2 \geqslant 9$. From (4.2), we thus have $x_1 \geqslant 3$. Applying Lemma 3.1, we deduce the inequality

$$y_2 - y_1 \geqslant 2((N+1)/2)^{x_1 - 1},$$

which in turn, together with (4.3), implies that

$$(N+1)^{x_1} > N + (N/3)^{\frac{\log(N)}{\log(N+1)}(2((N+1)/2)^{x_1-1}+1)}.$$

Since it is relatively easy to show that there are no solutions to this inequality with $N \geqslant 5$ and $x_1 \geqslant 3$, we conclude as desired.

Next, suppose that

$$(N+1)^{x_2} - N^{y_2} = (N+1)^{x_1} - N^{y_1} < 0,$$

so that $y_1 \geqslant 2$. Arguing as before, (2.2) yields

$$N^{y_1} > (N/3)^{x_2} + (N+1)^{x_1}. \tag{4.4}$$

Suppose first that $N = 4$. If $y_1 = 2$, we have $x_1 = 1$ and, from (4.4), since Lemma 3.1 implies $x_2 \equiv 1 \pmod 4$, that $x_2 = 5$. Since $4^{y_2} \neq 5^5 + 11 = 3136$, we reach a contradiction. It follows that $y_1 \geqslant 3$ and, since Lemma 3.1 gives $x_2 - x_1 \geqslant 4^{y_1 - 1}$, (4.4) yields

$$4^{y_1} > (4/3)^{4^{y_1-1}+x_1} + 5^{x_1}.$$

Since $x_1 \geqslant 1$ and $y_1 \geqslant 3$, this is a contradiction. If $N \geqslant 5$, Lemma 3.1 implies that

$$x_2 - x_1 \geqslant 2(N/2)^{y_1-1}$$

and so, from (4.4),

$$N^{y_1} > (N/3)^{2(N/2)^{y_1-1}+1} + N + 1.$$

This inequality contradicts $y_1 \geqslant 2$ and $N \geqslant 5$, completing the proof of Theorem 1.1. □

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