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ON THE REPRESENTATION OF UNITY BY BINARY CUBIC FORMS

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Abstract. If $F(x,y)$ is a binary cubic form with integer coefficients such that $F(x,1)$ has at least two distinct complex roots, then the equation $F(x,y)=1$ possesses at most ten solutions in integers x and y, nine if F has a nontrivial automorphism group. If, further, $F(x,y)$ is reducible over $\mathbb{Z}[x,y]$, then this equation has at most 2 solutions, unless $F(x,y)$ is equivalent under $GL_2(\mathbb{Z})$-action to either $x(x^2-xy-y^2)$ or $x(x^2-2y^2)$. The proofs of these results rely upon the method of Thue-Siegel as refined by Evertse, together with lower bounds for linear forms in logarithms of algebraic numbers and techniques from computational Diophantine approximation. Along the way, we completely solve all Thue equations $F(x,y)=1$ for F cubic and irreducible of positive discriminant $D_F \leq 10^6$. As corollaries, we obtain bounds for the number of solutions to more general cubic Thue equations of the form $F(x,y)=m$ and to Mordell's equation $y^2=x^3+k$, where m and k are nonzero integers.

1. Introduction

In 1909, Thue [57] derived the first general sharpening of Liouville's theorem on rational approximation to algebraic numbers, proving, if θ is algebraic of degree $n \geq 3$ and $\epsilon > 0$, that there exists a constant $c(\theta,\epsilon)$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c(\theta,\epsilon)}{q^{\frac{n}{2}+1+\epsilon}}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. It follows almost immediately, if $F(x,y)$ is an irreducible binary form (in $\mathbb{Z}[x,y]$) of degree at least three and m a nonzero integer, that the equation

$$F(x,y) = m \tag{1.1}$$

possesses at most finitely many solutions in integers x and y (to see this, apply Thue's inequality to the roots of $F(x,1)=0$). In the intervening years, there has developed an extensive body of literature devoted to explicitly solving "Thue" equations, or bounding the number of such integral solutions; in the latter regard, we mention a result of Bombieri and Schmidt [14] (see Stewart [53] for further refinements):

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Theorem 1.1. *If F is an irreducible binary form of degree n and m is a nonzero integer, then equation* (1.1) *possesses at most*

$$c\, n^{1+\omega(m)}$$

solutions in coprime integers x and y, where $\omega(m)$ denotes the number of distinct prime factors of m and the constant c is absolute (for n sufficiently large, one may take $c = 430$).

Note that this upper bound upon the number of solutions to (1.1) is independent of the coefficients of the form F; a result of this flavour was first deduced in 1983 by Evertse [23]. In a certain sense, this result is sharp, at least up to the constant c. Indeed, the example

$$F(x, y) = x^n + r(x - y)(2x - y) \cdots (nx - y),$$

where r is a nonzero integer, has the corresponding solutions $(1, 1), (1, 2), \ldots , (1, n)$ to (1.1) with $m = 1$.

The effective solution of an arbitrary Thue equation has its origins in the following theorem of Baker [4]:

Theorem 1.2. *If F is an irreducible binary form of degree n, $\kappa > n + 1$ and m is a nonzero integer, then every integer solution (x, y) of equation* (1.1) *satisfies*

$$\max\{|x|, |y|\} < c\, e^{\log^{\kappa} |m|}$$

where c is an effectively computable constant depending only on n, κ and the coefficients of F.

More recent refinements of this result, together with techniques from computational Diophantine approximation, have led to practical algorithms for solving Thue equations. We will discuss these briefly in Section 9; the reader is directed to [60] and [51] for more details.

In what follows, we restrict our attention to binary cubic forms with integer coefficients, i.e. polynomials of the shape

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

We require some terminology before we can state our results. Let us call forms F_1 and F_2 *equivalent* (and write $F_1 \sim F_2$) if they are equivalent under $GL_2(\mathbb{Z})$-action (i.e. if there exist integers a_1, a_2, a_3 and a_4 such that

$$F_1(a_1x + a_2y, a_3x + a_4y) = F_2(x, y)$$

for all x and y, where $a_1a_4 - a_2a_3 = \pm 1$). The *discriminant* D_F of such a form is given by

$$D_F = 18abcd + b^2c^2 - 27a^2d^2 - 4ac^3 - 4b^3d = a^4 \prod_{i<j} (\alpha_i - \alpha_j)^2$$

where α_1, α_2 and α_3 are the roots of the polynomial $F(x, 1)$. We denote by N_F the number of solutions in integers x and y of the Diophantine equation

$$F(x, y) = 1. \tag{1.2}$$

Note that if $F_1 \sim F_2$, then $N_{F_1} = N_{F_2}$ and $D_{F_1} = D_{F_2}$. The quantity N_F will be the primary object of study in this paper; its behaviour differs quite substantially depending on whether D_F is positive or negative. We discuss these situations in turn.

1.1. **Forms of negative discriminant.** When F is a binary cubic form of negative discriminant, we have a fairly complete understanding of N_F. In fact, more than 75 years ago, Delone [20] and Nagell [40] established independently that equation (1.2) has at most five solutions if $D_F < 0$. More precisely, they proved

Theorem 1.3. *If F is an irreducible binary cubic form with integer coefficients and $D_F < 0$, then $N_F \leq 5$. Moreover, if $N_F = 5$, then F is equivalent to*

$$x^3 - xy^2 + y^3,$$

with $D_F = -23$ and, if $N_F = 4$, then F is equivalent to either

$$x^3 + xy^2 + y^3 \text{ or } x^3 - x^2y + xy^2 + y^3,$$

with discriminant -31 or -44, respectively.

Their proofs rely crucially upon the fact that, if $D_F < 0$, the number field generated over $\mathbb{Q}$ by the real root of the equation $F(x,1) = 0$ has a ring of integers with a single fundamental unit. They utilize what would now be considered to be a special case of Skolem's p-adic method (though, in the interests of historical fairness, it might be reasonably regarded as the origin of this technique) together with what Delone terms an "ascent algorithm"; the reader is directed to [21], Chapter VI for details.

1.2. **Forms of positive discriminant.** The situation where $D_F > 0$ is complicated by the fact that the number field $\mathbb{Q}(\epsilon)$ (where ϵ is any real root of $F(x,1) = 0$) has a ring of integers generated by a pair of fundamental units. In principle, as noted by Ljunggren [33] and [34], the p-adic method used in case $D_F < 0$ may in fact be extended to treat this more difficult problem. For particular cubic forms (i.e. those with $D_F = 49, 81$; see e.g. [33] and [6]), such an approach has been employed to solve equation (1.2). It does not, however, appear to be a straightforward matter to derive an explicit upper bound upon N_F by this method, valid for arbitrary cubic forms of positive discriminant. The main reason for this is that, in order to apply the local method of Skolem, one requires exact information about fundamental units in certain quadratic extensions of $\mathbb{Q}(\epsilon)$.

In 1929, Siegel [48] used the theory of Padé approximation to binomial functions (via the hypergeometric function) to show, for F cubic of positive discriminant, that equation (1.2) has at most 18 solutions in integers x and y. Refining these techniques, Evertse [24] reduced this upper bound to 12. In fact, in 1949, Gel'man had already demonstrated that there could be at most 10 such solutions, provided the discriminant of the form was large enough (see Delone and Fadeev [21], Chapter V for a proof; Evertse [24] indicates that $D_F > 5 \times 10^{10}$ suffices).

The main result of this paper is, in essense, a technical appendix to [24]:

Theorem 1.4. *If $F(x, y)$ is a homogeneous cubic polynomial with integral coefficients for which $F(x, 1)$ has at least two distinct complex roots, then the equation*

$$F(x, y) = 1$$

possesses at most 10 solutions in integers x and y.

As observed by Bombieri and Schmidt [14], bounds for the number of solutions to equation (1.2) lead to corresponding results for equation (1.1). In fact, Theorem 1.4 immediately implies

Corollary 1.5. *If $F(x,y)$ is a homogeneous cubic polynomial with integral coefficients and nonzero discriminant and m is a nonzero integer, then the equation*

$$F(x,y) = m$$

possesses at most $10 \times 3^{\omega(m)}$ solutions in coprime integers x and y. Here, $\omega(m)$ denotes the number of distinct prime factors of m.

Additionally, a result of the flavour of Theorem 1.4 leads, via an argument of Mordell [39], to bounds for the number of solutions of Mordell's equation:

Corollary 1.6. *If k is a nonzero integer, then the equation*

$$y^2 = x^3 + k$$

has at most $10\ h_3(-108k)$ solutions in integers x and y, where $h_3(-108k)$ is the class number of binary cubic forms of discriminant k.

We note, if $\epsilon > 0$, one may show that

$$h_3(-108k) \ll |k|^{1/2+\epsilon}$$

(see e.g. the forthcoming paper of the author and T. D. Wooley [13]).

Theorem 1.4 is almost sharp, since the equation

$$x^3 - x^2y - 2xy^2 + y^3 = 1 \tag{1.3}$$

(corresponding to a cubic form of discriminant 49) has the nine integral solutions $(x,y) = (1,0),(0,1),(-1,1),(-1,-1),(2,1),(-1,-2),(5,-4),(4,9)$ and $(-9,-5)$. That this list is complete was stated by Ljunggren [33] and proved by Baulin [6] (via the p-adic method alluded to earlier). In the following table, we give representatives of all known equivalence classes of irreducible cubic forms for which $N_F \geq 6$. In each case, N_F has now been determined exactly (refer to the cited references):

$F(x,y)$	D_F	N_F	References
$x^3 - x^2y - 2xy^2 + y^3$	49	9	[6], [25], [33], [44]
$x^3 - 3xy^2 + y^3$	81	6	[25], [33], [59]
$x^3 - 4xy^2 + y^3$	229	6	[15], [25], [44]
$x^3 - 5xy^2 + 3y^3$	257	6	[25]
$x^3 + 2x^2y - 5xy^2 + y^3$	361	6	[25]

Presumably, the "truth" of the matter is the following conjecture (essentially due to Nagell [41] and refined by Pethő [42] and Lippok [32]), which states that the forms in the above table are, up to equivalence, the only irreducible cubics with $N_F \geq 6$:

Conjecture 1.7. *If F is a binary cubic form with $D_F > 0$, then $N_F = 9$ if $D_F = 49$, $N_F = 6$ if $D_F \in \{81, 229, 257, 361\}$ and $N_F \leq 5$ otherwise.*

As we describe in Section 4, there are infinitely many inequivalent cubic forms F for which $D_F > 0$ and $N_F = 5$.

Our proof of Theorem 1.4 consists of some (very) slight refinements of a number of technical lemmata from [24], together with some recent techniques from computational Diophantine approximation, based upon lower bounds for linear forms in logarithms of algebraic numbers and the L^3 lattice basis reduction algorithm. The paper is organized as follows. In Section 2, we treat the (rather simple) case of reducible forms. In such a situation, we are able to derive precise information regarding solutions to equation (1.2). In Section 3, we consider irreducible forms with nontrivial automorphism groups, obtaining a (sharp) refinement of Theorem 1.4. Section 4 contains mostly historical, expository remarks about families of cubic Thue equations and their effective solution. In Section 5, we begin the proof of Theorem 1.4 for irreducible forms, following the classical reduction theory of cubic forms of positive discriminant. Section 6 introduces the Padé approximants that play a crucial role in this proof and provides a number of fundamental inequalities concerning them. In Section 7, we apply these inequalities to complete the proof of Theorem 1.4 for cubic forms F with discriminant $D_F > 24000$. In Section 8, we discuss the problem of finding representatives for each equivalence class of cubic forms with positive discriminant below a certain bound. Finally, in Section 9, we briefly describe how one solves a collection of cubic Thue equations, via lower bounds for linear forms in logarithms of algebraic numbers, together with techniques from computational Diophantine approximation, and present the results of these computations in our situation, completing the proof of Theorem 1.4.

2. Reducible forms

Let us take a brief interlude from the principal matter at hand to discuss the (much simpler) situation where the form $F(x, y)$ is reducible over $\mathbb{Z}[x, y]$. In general, equation (1.2) may have infinitely many integral solutions; $F(x, y)$ could, for instance, be a power of a linear or indefinite binary quadratic form that represents unity. If $F(x, y)$ is a reducible cubic form, however, we may very easily derive a stronger version of Thue's theorem, under the assumption that $F(x, 1)$ has at least two distinct zeros. Indeed, we have

Theorem 2.1. *Suppose that $F(x, y)$ is a reducible cubic form such that $F(x, 1)$ has at least two distinct roots (over $\mathbb{C}$). If $D_F > 0$, then we have $N_F \leq 4$. Further, if $N_F = 4$, then F is equivalent to*

$$x\left(x^2 - xy - y^2\right),$$

whereby $D_F = 5$, and if $N_F = 3$, then F is equivalent to

$$x\left(x^2 - 2y^2\right)$$

(with $D_F = 32$). If, on the other hand, $D_F \leq 0$, then $N_F \leq 2$ and $N_F = 2$ implies that F is equivalent to either

$$x\left(x^2 + nxy + ny^2\right)$$

for some n with $1 \leq n \leq 3$ (corresponding to $D_F = -3, -16$ or -27, respectively), or to

$$x\,(x + y)^2 \quad \text{or} \quad x\,(x + 2y)^2$$

(with $D_F = 0$).

Proof. Let us suppose that $F(x, y)$ is a binary cubic form, reducible over $\mathbb{Z}[x, y]$, and that $F(x, 1)$ has at least two distinct complex roots. We begin by applying the following (almost trivial) lemma:

Lemma 2.2. *If F is a reducible binary cubic form, then equation* (1.2) *has a solution in integers x and y precisely when $F \sim G$ for a form G satisfying*

$$G(x, y) = x\left(x^2 + axy + by^2\right) \tag{2.1}$$

where $a, b \in \mathbb{Z}$.

Proof of Lemma 2.2. If $F \sim G$ and G satisfy (2.1), then $G(1,0)=1$ and so $N_F = N_G \geq 1$. Conversely, if there exist integers x_0 and y_0 such that $F(x_0,y_0)=1$, then necessarily $\gcd(x_0,y_0)=1$ and so there exist integers a and b with $ax_0+by_0=1$. It follows that the map $x \to ax+by$, $y \to -y_0x+x_0y$ has determinant 1 and sends (x_0,y_0) to $(1,0)$, whereby $F \sim \overline{F}$, for

$$\overline{F}(x,y) = (x+jy)\left(x^2+kxy+ly^2\right)$$

and $j,k,l \in \mathbb{Z}$. The lemma follows upon noting that the map $x \to x-jy$, $y \to y$ has determinant 1. □

It follows that F must be equivalent to a form G satisfying (2.1), with at least one of a or b nonzero (since otherwise $G(x,1)$ and hence $F(x,1)$ would have only a single root). We readily compute that

$$D_G = b^2\left(a^2-4b\right). \tag{2.2}$$

If $G(x,y)=1$ for x and y integral, then it follows that $x=\pm 1$. In case $x=1$, then substituting for x in (2.1), we find that

$$by^2+ay=0. \tag{2.3}$$

Similarly, if $x=-1$, then (2.1) yields

$$by^2+ay+2=0. \tag{2.4}$$

Suppose first that $a=0$ (so that $b\neq 0$). If $x=1$, then (2.3) leads to only the solution $(x,y)=(1,0)$. On the other hand, if $x=-1$, from (2.4) we have $y=\pm\sqrt{-2/b}$. In order to have $y\in\mathbb{Z}$, we require that $b=-2$ (so that $y=\pm 1$). It follows that if $a=0$, then $N_G=1$ unless

$$G(x,y) = x\left(x^2-2y^2\right).$$

Noting that the equation $x(x^2-2y^2)=1$ has the integral solutions $(x,y) = (1,0),(-1,1)$ and $(-1,-1)$, we conclude that $N_G=3$.

Next, let us suppose that a is nonzero. If $D_F=0$, then $F\sim G$ where, from (2.2), we have either

$$G(x,y) = x^2\left(x+ay\right)$$

or

$$G(x,y) = x\left(x+\left(\frac{a}{2}\right)^2 y\right)^2.$$

In the first case, if $G(x,y)=1$, we have $x=\pm 1$ and $x+ay=1$, whereby $(x,y) = (1,0)$ or $ay=2$ (so that $a\in\{\pm 1,\pm 2\}$). We may readily show that such forms are equivalent, under $GL_2(\mathbb{Z})$-action, to either $x(x+y)^2$ or $x(x+2y)^2$. The second possibility for G leads to the same conclusion. If, however, both a and b (and hence D_F and D_G) are nonzero, we may note that equations (2.3) and (2.4) lead to

$$y \in \left\{0, \frac{-a}{b}, \frac{-a\pm\sqrt{a^2-8b}}{2b}\right\}.$$

Since these four values sum to zero, if $N_G\geq 3$, we necessarily have $N_G=4$. In this case,

$$a\equiv 0 \pmod b \text{ and } a^2-8b=m^2$$

for some $m \in \mathbb{Z}$, and also

$$-a \pm m \equiv 0 (\operatorname{mod} b),$$

whence $m \equiv 0 (\operatorname{mod} b)$. Combining these facts, it follows that b^2 divides $8b$ and so, since $b \neq 0$ by hypothesis, $b \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. Now

$$a \equiv m \equiv 0 (\operatorname{mod} b) \text{ and } a^2 - m^2 = 8b,$$

and so there exist positive integers u and v with $u^2 - v^2 = 8/b$ which implies that $b = \pm 1$. If $b = 1$, then $a = \pm 3$, while $b = -1$ leads to $a = \pm 1$. In all four cases, we find that $D_G = 5$ and so, since there is a unique equivalence class of binary quadratic (and hence of reducible cubic) forms of discriminant 5 ($\mathbb{Q}(\sqrt{5})$ has class number 1), it follows that

$$G(x, y) \sim x \left(x^2 - xy - y^2 \right).$$

Observing that the equation $x(x^2 - xy - y^2) = 1$ has the solutions $(x, y) = (1, 0), (1, -1), (-1, -1)$ and $(-1, 2)$ completes our analysis in the cases where $N_F \geq 3$.

Finally, assume that $D_F < 0$ and $N_F = 2$. From (2.2), we have $a^2 < 4b$ and so, in order to have two distinct integral solutions to $G(x, y) = 1$, we must have $\frac{a}{b} \in \mathbb{Z} \setminus \{0\}$. It follows that

$$\left(\frac{a}{2} \right)^2 < b \leq |a| \tag{2.5}$$

and so $1 \leq |a| \leq 3$. In each of these cases, we find from (2.5) that $|a| = b$ and $D_G = -3, -16$ or -27, corresponding to $b = 1, 2$ or 3, respectively. Again noting that $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ have class number 1, this concludes the proof of Theorem 2.1. □

It is worth observing that there are infinite classes of nonequivalent reducible cubic forms F with $D_F > 0$ and $N_F = 2$ (in contrast to the case of negative discriminants detailed in the preceding Theorem). Indeed, the forms

$$F_n(x, y) = x \left(x^2 + nxy - ny^2 \right)$$

have positive discriminant if $n \geq 1$ or $n \leq -5$, and $F_n(x, y) = 1$ for $(x, y) = (1, 0)$ and $(1, 1)$.

An amusing corollary of Theorem 2.1 is the following irreducibility criterium (where we write, for a polynomial $P(x)$ of degree n, $P^*(x, y)$ for the corresponding binary form, given by $P^*(x, y) = y^n P(x/y)$):

Corollary 2.3. *Suppose that $P(x)$ is a cubic polynomial with integral coefficients and discriminant D_P, for which the equation $P^*(x, y) = 1$ has at least three distinct solutions in integers x and y. Then, if $D_P \notin \{5, 32\}$, either $P(x)$ is irreducible over $\mathbb{Z}[x]$ or there exist coprime integers p and q such that $P(x) = (px + q)^3$.*

In particular, this implies the irreducibility of polynomials of the shape

$$P(x) = r (x - a_1)(x - a_2)(x - a_3) \pm 1,$$

if r is a nonzero integer and a_1, a_2 and a_3 are distinct integers, or if $r = \pm 1$ and at least two of a_1, a_2 and a_3 are distinct (unless, without loss of generality, we have $a_1 = a_2 = a_3 \pm 2$). See also Schur [45] and Heuberger [28].

3. Automorphisms of cubic forms

For a given binary form F, let Aut_F denote the subgroup of elements of $SL_2(\mathbb{Z})$ which fix F (i.e. $\sigma \in Aut_F$ implies $\sigma(F) = F$). The following result of Ayad [3] (see also Proposition 2.12 of Shintani [46]) completely characterizes those cubic forms for which Aut_F is nontrivial:

Theorem 3.1. *If F is a binary cubic form with $D_F \neq 0$, then Aut_F is either trivial or cyclic of order 3. The latter case occurs precisely for those forms which are equivalent under $SL_2(\mathbb{Z})$-action to*

$$G_{m,n}(x, y) = mx^3 + nx^2y - (n + 3m)xy^2 + my^3$$

for some $m, n \in \mathbb{Z}$ (whereby $D_F = (n^2 + 3mn + 9m^2)^2$). Further, if $|Aut_F| = 3$, then $N_F \equiv 0 (\bmod 3)$.

Applying Theorem 1.4 immediately yields the following:

Corollary 3.2. *If F is a binary cubic form for which Aut_F is nontrivial and D_F is nonzero, then $N_F \leq 9$.*

As previously noted, the equation $G_{1,-1}(x, y) = 1$ (i.e. equation (1.3)) possesses nine integral solutions and thus this result is sharp. Since the forms $G_{1,r}(x, y)$ have $N_G \geq 3$ (indeed, as we mention in Section 4, N_G has been completely determined for these forms), it might be tempting to suppose that these are, up to equivalence, the only forms with nontrivial automorphism groups for which (1.2) is solvable. This is, in fact, untrue as there are infinite families of indices (m, n) for which

$$G_{m,n} \not\sim G_{1,r} \tag{3.1}$$

for all $r \in \mathbb{Z}$, but still $N_{G_{m,n}} \geq 3$. By way of example, if $(m, n) = (2k+1, -3k-1)$ for $k \in \mathbb{Z}$, then $G_{m,n}(2, 1) = 1$ and (3.1) holds for infinitely many values of k. To see this, observe that $D_{G_{m,n}} = (27k^2 + 27k + 7)^2$ and so, if $G_{m,n} \sim G_{1,r}$, then

$$27k^2 + 27k + 7 = r^2 + 3r + 9$$

or

$$(2r + 3)^2 - 27(2k + 1)^2 = -26,$$

whose solutions grow exponentially in r and k. Note that if $N_{G_{m,n}} > 0$, parity considerations ensure, necessarily, that m is odd.

4. Families of cubic Thue equations

The first infinite parametrized families of Thue equations to be solved were, fittingly enough, done so by Thue [58], a fact that seems to be frequently overlooked. Indeed, a proof that the equation

$$(a + 1)x^n - ay^n = 1 \tag{4.1}$$

has only the solution $x = y = 1$ in positive integers, for a suitably large in relation to prime $n \geq 3$, follows immediately from the main theorem of [58]. As a specific example, in Beispiel 1 of [58], one finds explicit bounds for integral solutions of the inequality

$$\left|(a + 1)x^3 - ay^3\right| \leq c$$

with $a \geq 37$. These readily imply that the equation

$$(a+1)x^3 - ay^3 = 1$$

has only the solution $x = y = 1$ in integers, provided $a \geq 386$ (or, with a modicum of computation, with $a \geq 37$) (see also Siegel [49]). We remark that the author [11], building on the author's joint paper with de Weger [12], has recently extended Thue's result to show that if a and n are arbitrary positive integers with $n \geq 3$, then equation (4.1) has only the solution $x = y = 1$ in positive integers.

In the case of parametrized families of cubic Thue equations of positive discriminant, results are of a much more recent nature. In 1990, using the fact that the underlying number fields are the so-called "simplest cubics", Thomas [55] was able to show that the equations

$$G_{1,n}(x,y) = x^3 + nx^2y - (n+3)xy^2 + y^3 = 1$$

have only the solutions $(1,0), (0,1)$ and $(-1,-1)$ in integers, provided $n \geq 1.365 \times 10^7$. This restriction was later removed by Mignotte [36] (except for the equations with $n = -1, 0$ or 2, in which case, as previously observed, we have $6, 3$ and 3 additional solutions). For a good overview of families of Thue equations (cubic and otherwise) that have been solved in recent years, the reader is directed to Heuberger [29]. For the purposes of the paper at hand, we will mention two further families, the only ones known for which $N_F \geq 5$. It seems not unlikely that there are other infinite families of cubic forms with $N_F = 5$.

Define $F_m(x,y)$ and $G_n(x,y)$ by

$$F_m(x,y) = x^3 - (m+1)x^2y + mxy^2 + y^3$$

and

$$G_n(x,y) = x^3 - n^2x^2y + y^3,$$

for $m, n \in \mathbb{Z}$. Provided $m \neq -2, -1, 0$ or 1, the equation $F_m(x,y) = 1$ has the five distinct integral solutions $(x,y) = (1,0), (1,1), (1,-m-1), (0,1)$ and $(m,1)$. That this list is complete was proven, independently, by Lee [31] and Mignotte and Tzanakis [38], for m suitably large (and, recently, by Mignotte [37], for $m \geq 2$). One observes that the cases $m = 0$ and $m = 1$ correspond to discriminants -23 and -31, respectively, i.e. to extreme examples of forms of negative discriminant. Similarly, if $n \neq 0, 1$ or -1, then $G_n(x,y) = 1$ has the distinct solutions $(x,y) = (1,0), (1,n), (1,-n), (0,1)$ and $(n^2,1)$. As Pethő [42] (Theorem 3) has remarked, the families F_m and G_n are essentially disjoint. Indeed, if $F_m \sim G_n$ for some integers m and n, then $D_{F_m} = D_{G_n}$. Since $D_{F_m} = m^4 + 2m^3 - 5m^2 - 6m - 23$ and $D_{G_n} = 4n^6 - 27$, we therefore have

$$4n^6 = m^4 + 2m^3 - 5m^2 - 6m + 4$$

or, equivalently,

$$\left(m^2 + m - 3\right)^2 - \left(2n^3\right)^2 = 5.$$

It follows that $m^2 + m - 3 = \pm 3$ and $2n^3 = \pm 2$, whence $m \in \{-3, -1, 0, 2\}$ and $n \in \{-1, 1\}$. In each of these cases, the relevant form is equivalent to $x^3 - xy^2 + y^3$, of discriminant -23.

Observe that both families F_m and G_n are equivalent to

$$F(x,y) = (x - a_1y)(x - a_2y)(x - a_3y) \pm y^3, \tag{4.2}$$

for suitable choices of integers a_1, a_2 and a_3. Such polynomials have been termed *split forms* by Thomas [55] (see also [56]). There has been substantial research on split families in recent years; the reader is directed to [29] for details and extensive references. We note that, in Section 9, we will discuss a family of non-split forms for which $N_F \geq 4$ (see also [32]).

5. Reduction to a diagonal form

We return now to consideration of the problem of bounding N_F for an arbitrary cubic form of positive discriminant. A key observation that enables us to derive relatively precise results in this situation is that we may reduce the problem at hand to consideration of a diagonal form over a suitable imaginary quadratic field. The method of Thue-Siegel is particularly well suited for application to such forms. Virtually all of what follows is classical, deriving from work of Eisenstein, Hermite, Arndt and Berwick; the reader is directed to Dickson [22] (Vol. 3, Chapter 12) for references.

Let us define, for a cubic form F, an associated quadratic form, the Hessian $H = H_F$, and a cubic form $G = G_F$, by

$$H(x,y) = -\frac{1}{4}\left(\frac{\delta^2 F}{\delta x^2}\frac{\delta^2 F}{\delta y^2} - \left(\frac{\delta^2 F}{\delta x \delta y}\right)^2\right)$$

and

$$G(x,y) = \frac{\delta F}{\delta x}\frac{\delta H}{\delta y} - \frac{\delta F}{\delta y}\frac{\delta H}{\delta x}.$$

These forms are covariant with respect to the action of $GL_2(\mathbb{Z})$; i.e.

$$H_{F\circ\gamma} = H_F \circ \gamma \text{ and } G_{F\circ\gamma} = G_F \circ \gamma$$

for all $\gamma \in GL_2(\mathbb{Z})$. If we write

$$F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

and

$$H(x,y) = Ax^2 + Bxy + Cy^2,$$

then it follows by routine calculation that

$$A = b^2 - 3ac,\ B = bc - 9ad,\ C = c^2 - 3bd$$

and

$$B^2 - 4AC = -3D_F.$$

Further, these forms are related to $F(x,y)$ via the identity

$$4H(x,y)^3 = G(x,y)^2 + 27D_F F(x,y)^2 \tag{5.1}$$

(see Theorem 1 of Mordell [39], Chapter 24). Following Hermite, we will call a cubic form F *reduced* if the Hessian of F, $H(x,y) = Ax^2 + Bxy + Cy^2$, satisfies

$$C \geq A \geq |B|.$$

It is a basic fact that every cubic form F of positive discriminant is equivalent to a reduced form. The notion of reduction here is classical and differs somewhat from that used by Belabas and Cohen in [8] and [9]. In fact, it is this latter, more stringent version of reduction we will utilize in Section 8.

We begin with a lemma that characterizes "very small" solutions of equation (1.2).

Lemma 5.1. *If F is an irreducible, reduced binary cubic form with positive discriminant D_F and Hessian H, then the equation $F(x,y)=1$ has at most one solution in integers x and y with $H(x,y)<\frac{1}{2}\sqrt{3D_F}$. If this solution exists, it is given by one of $(x,y)=(\pm 1,0)$.*

Proof. Suppose that (x,y) is a solution to $F(x,y)=1$ with $y\neq 0$. If $|y|\leq |x|$, then, since $A\geq |B|$, we have that

$$H(x,y)\geq Cy^2\geq C\geq \frac{1}{2}\sqrt{3D_F},$$

where the last inequality readily follows from $B^2-4AC=-3D_F$. If, on the other hand, $|y|\geq |x|+1$, then

$$H(x,y)\geq (C-|B|)\,y^2+|B||y|+Ax^2.$$

Since this is an increasing function of $|y|$ and $y\neq 0$, we have

$$H(x,y)\geq C+Ax^2\geq C\geq \frac{1}{2}\sqrt{3D_F}.$$

We conclude, if $H(x,y)<\frac{1}{2}\sqrt{3D_F}$, that $y=0$ (and so $x=\pm 1$ accordingly). $\square$

When one speaks of "irreducible, reduced forms", as Davenport comments, "the terminology is unfortunate, but can hardly be avoided" ([18], page 184).

From now on, let us write $\Delta=3D_F$ and assume $D_F\geq 24000$ (whence $\Delta\geq 72000$). We will work in the number field $M=\mathbb{Q}(\sqrt{-\Delta})$ for a fixed choice of the square root. From (5.1),

$$\frac{G(x,y)\pm 3\sqrt{-\Delta}F(x,y)}{2}$$

are cubic forms in $M[x,y]$ with coefficients conjugate to one another and no common factors (since $F(x,y)$ is also irreducible over M). It follows that they are cubes of linear forms over M with the same properties, say $\xi(x,y)$ and $\eta(x,y)$, where (see Evertse [24], displayed equation (11))

$$\xi(x,y)^3-\eta(x,y)^3=3\sqrt{-\Delta}F(x,y),$$

$$\xi(x,y)^3+\eta(x,y)^3=G(x,y),$$

$$\xi(x,y)\eta(x,y)=H(x,y)$$

and

$$\frac{\xi(x,y)}{\xi(1,0)}\text{ and }\frac{\eta(x,y)}{\eta(1,0)}\in M[x,y].$$

We call a pair of forms ξ and η satisfying the above properties a pair of *resolvent forms*, and note that if (ξ,η) is one pair, there are precisely two others, given by $(\omega\xi,\omega^2\eta)$ and $(\omega^2\xi,\omega\eta)$ for ω a primitive cube root of unity. We say that a pair of rational integers (x,y) is *related to a pair of resolvent forms* (ξ,η) if

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right|=\min_{0\leq k\leq 2}\left|e^{2k\pi i/3}-\frac{\eta(x,y)}{\xi(x,y)}\right|. \tag{5.2}$$

We will first derive an upper bound for

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right|,$$

following an argument of Evertse [24]. From our definitions, we have

$$\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right| = \frac{3\sqrt{\Delta}}{|\xi(x,y)|^3}$$

and will, in consequence of Lemma 5.1, assume $H(x,y) \geq \sqrt{\Delta}/2$, whereby

$$|\xi(x,y)| \geq \frac{1}{\sqrt{2}}\Delta^{1/4}.$$

It follows from $\Delta \geq 72000$ that

$$\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right| \leq \frac{6\sqrt{2}}{\Delta^{1/4}} < 0.519. \tag{5.3}$$

Since $\eta(x,y)/\xi(x,y)$ has modulus one, equation (5.2) implies, if we write $\theta = \arg\left(\frac{\eta(x,y)}{\xi(x,y)}\right)$, that $3\theta = \arg\left(\frac{\eta(x,y)^3}{\xi(x,y)^3}\right)$. By virtue of (5.3),

$$2-2\cos(3\theta) < (0.519)^2,$$

and so $|\theta| < 0.176$. Since

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right| \leq \left|\arg\left(\frac{\eta(x,y)}{\xi(x,y)}\right)\right| = \frac{1}{3}\left|\arg\left(\frac{\eta(x,y)^3}{\xi(x,y)^3}\right)\right|,$$

we have

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right| \leq \frac{1}{3}\frac{3|\theta|}{\sqrt{2-2\cos(3\theta)}}\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|,$$

whence applying our upper bound for $|\theta|$ implies

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{1.012\sqrt{\Delta}}{|\xi(x,y)|^3}. \tag{5.4}$$

This inequality will enable us to obtain our first "gap" principle; i.e. a result that prevents "suitably large" solutions to (1.2) from lying too close together. Suppose that we have distinct solutions to (1.2), related to (ξ,η) and indexed by i, say (x_i,y_i), with $|\xi(x_{i+1},y_{i+1})| \geq |\xi(x_i,y_i)|$). For concision, we will write $\eta_i = \eta(x_i,y_i)$ and $\xi_i = \xi(x_i,y_i)$. Since $\xi(x,y)\eta(x,y) = H(x,y)$ is a quadratic form of discriminant $-\Delta$, it follows that

$$\xi_2\eta_1 - \xi_1\eta_2 = \pm\sqrt{-\Delta}\,(x_1y_2 - x_2y_1)$$

and, since $(x_1,y_1),(x_2,y_2)$ are distinct solutions to $F(x,y)=1$, we have

$$\sqrt{\Delta} \leq |\xi_2\eta_1 - \xi_1\eta_2| \leq |\xi_1||\xi_2|\left(\left|1-\frac{\eta_1}{\xi_1}\right| + \left|1-\frac{\eta_2}{\xi_2}\right|\right).$$

Thus (5.4) implies

$$\sqrt{\Delta} < |\xi_1||\xi_2|(1.012\sqrt{\Delta})\left(|\xi_1|^{-3} + |\xi_2|^{-3}\right).$$

It follows that

$$|\xi_1|^{-3} + |\xi_2|^{-3} > 0.988\,|\xi_1|^{-1}|\xi_2|^{-1}$$

and so

$$|\xi_1|^3 + |\xi_2|^3 > 0.988\ |\xi_1|^2|\xi_2|^2.$$

If we write $|\xi_2| = \alpha|\xi_1|^2$, then

$$|\xi_1|^3 + \alpha^3|\xi_1|^6 > 0.988\ \alpha^2|\xi_1|^6$$

whence

$$\left(0.988\ \alpha^2 - \alpha^3\right)|\xi_1|^3 < 1.$$

Since, by assumption,

$$|\xi_1| > \frac{1}{\sqrt{2}}\Delta^{1/4} > 11.582,$$

the above inequality implies that $\alpha > 0.987$; i.e. we have proven

Lemma 5.2. *If (x_1, y_1) and (x_2, y_2) are distinct solutions to (1.2), related to (ξ, η), with*

$$|\xi(x_2, y_2)| \geq |\xi(x_1, y_1)| \geq \frac{1}{\sqrt{2}}\Delta^{1/4}$$

and $\Delta \geq 72000$, then

$$|\xi(x_2, y_2)| > 0.987\ |\xi(x_1, y_1)|^2.$$

Let us now assume that there are 4 distinct solutions to (1.2) related to a pair of resolvent forms (ξ, η), corresponding to ξ_{-1}, ξ_0, ξ_1 and ξ_2, where we have ordered these in nondecreasing modulus. We will deduce a contradiction, implying that at most 3 such solutions can exist, which, with Lemma 5.1, will prove Theorem 1.4. Two applications of Lemma 5.2 imply

$$|\xi_1| > (0.987)^3|\xi_{-1}|^4$$

and, since

$$|\xi_{-1}| \geq \frac{1}{\sqrt{2}}\Delta^{1/4},$$

we may conclude that

$$|\xi_1| > 0.24\ \Delta. \tag{5.5}$$

This inequality will prove crucial in establishing stronger gap principles in the next section.

6. Approximating polynomials

To prove Theorem 1.4 for forms of large discriminant, we apply arguments due to Siegel [48], [49] and [50], with refinements by Gel'man (see Delone and Fadeev [21]) and Evertse [24]. We note that the method employed actually leads to bounds on the number of solutions to the Diophantine inequality

$$|F(x, y)| \leq c,$$

though we will not pursue this here (see [24] for details). Following Evertse (see also Thue [58], Siegel [48] and Delone and Fadeev [21]), we define polynomials

$$A_{r,g}(z) = \sum_{m=0}^{r} \binom{r-g+1/3}{m}\binom{2r-g-m}{r-g}(-z)^m$$

and

$$B_{r,g}(z) = \sum_{m=0}^{r-g} \binom{r-1/3}{m} \binom{2r-g-m}{r-g} (-z)^m$$

for $r \in \mathbb{N}$ and $g \in \{0,1\}$. We have

$$A_{r,g} = \binom{2r-g}{r} F\left(-\frac{1}{3} - r + g, -r, -2r+g, z\right)$$

and

$$B_{r,g} = \binom{2r-g}{r} F\left(\frac{1}{3} - r, -r+g, -2r+g, z\right),$$

where

$$F(\alpha,\beta,\gamma,z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} z^n$$

is the standard hypergeometric function, satisfying the differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (\gamma - (1+\alpha+\beta)z)\,\frac{dF}{dz} - \alpha\beta F = 0.$$

$A_{r,g}(z)$ and $B_{r,g}(z)$ are essentially diagonal Padé approximants to the binomial function $(1-z)^{1/3}$, defined by

$$(1-z)^{1/3} = \sum_{m=0}^{\infty} \binom{1/3}{m} (-z)^m,$$

for $|z| < 1$. In fact, we have (see Lemma 3(i) of [24])

$$A_{r,g}(z) - (1-z)^{1/3} B_{r,g}(z) = z^{2r+1-g} F_{r,g}(z), \tag{6.1}$$

where $F_{r,g}(z)$ is a power series with rational coefficients.

We note that the polynomials $A_{r,g}$ and $B_{r,g}$ have coefficients that, generally speaking, possess large integer common factors; we will exploit this fact for small values of the parameter r. To be precise, let us define $C_{r,g}$ to be the greatest common divisor of the numerators of the coefficients of $A_{r,g}(z)$ (or, equivalently, from (6.1), the greatest common divisor of the numerators of the coefficients of $B_{r,g}(z)$). In general, we may show that

$$\lim_{r\to\infty} C_{r,g}^{1/r} = 3\sqrt{3}\, e^{-\frac{\pi\sqrt{3}}{6}} \sim 2.09807\ldots$$

(see e.g. [10]), but, for our purposes, it suffices to observe that $C_{r,g}$ is as follows for certain pairs (r,g) with $r \leq 8$:

(6.2)

(r,g)	$C_{r,g}$	(r,g)	$C_{r,g}$	(r,g)	$C_{r,g}$
$(1,1)$	1	$(4,0)$	5	$(7,1)$	4
$(1,0)$	2	$(5,0)$	28	$(7,0)$	88
$(2,0)$	1	$(6,1)$	14	$(8,1)$	11
$(3,0)$	20	$(6,0)$	14	$(8,0)$	55

We consider the complex sequences $\Sigma_{r,g}$ given by

$$\Sigma_{r,g} = \frac{\eta_2}{\xi_2} A_{r,g}(z_1) - \frac{\eta_1}{\xi_1} B_{r,g}(z_1)$$

where $z_1 = 1 - \eta_1^3/\xi_1^3$. Note that

$$|z_1| = \frac{3\sqrt{\Delta}}{|\xi_1|^3} < 218\ \Delta^{-5/2} < 10^{-9},$$

where the inequalities follow from (5.5) and $\Delta \geq 72000$. Let us define

$$\Lambda_{r,g} = \frac{1}{C_{r,g}}\ \xi_1^{3r+1-g}\xi_2\ \Sigma_{r,g}.$$

The key observation to make is that $\Lambda_{r,g}$ is either an integer in $M = \mathbb{Q}(\sqrt{-\Delta})$ or a cube root of such an integer. If $\Lambda_{r,g} \neq 0$, this provides a lower bound on $|\Lambda_{r,g}|$. In fact, from the proof of Lemma 5 of Evertse [24], we have

$$\Lambda_{r,0} \in \sqrt{-\Delta}\ \mathbb{Z}$$

and

$$\Lambda_{r,1}^3 \in \mathcal{O}_M \setminus \mathbb{Z},$$

where $\mathcal{O}_M$ denotes the ring of integers of M; i.e.

$$\mathcal{O}_M = \left\{ \frac{m + n\sqrt{-\Delta}}{2} : m, n \in \mathbb{Z}, m \equiv n\Delta (\mathrm{mod}\, 2) \right\}.$$

If $\Lambda_{r,g} \neq 0$, it follows that

$$|\Lambda_{r,0}| \geq \sqrt{\Delta}$$

and

$$\left|\Lambda_{r,1}^3\right| \geq \frac{1}{2}\sqrt{\Delta},$$

whereby

$$|\Lambda_{r,g}| \geq 2^{-g/3}\Delta^{1/2-g/3} \tag{6.3}$$

for $g \in \{0, 1\}$. To obtain an upper bound for $|\Lambda_{r,g}|$, we appeal to estimates of Evertse [24] for $|A_{r,g}(z_1)|$ and $|F_{r,g}(z_1)|$:

Lemma 6.1. *Let r and g be integers with $r \geq 1$, $g \in \{0, 1\}$ and $z \in \mathbb{C}$.*
(i) If $|z| < 1$, then

$$|F_{r,g}(z)| \leq \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}} (1 - |z|)^{-\frac{1}{2}(2r+1-g)}.$$

(ii) If $|1 - z| \leq 1$, then

$$|A_{r,g}(z)| \leq \binom{2r-g}{r}.$$

If we combine inequality (6.3) with Lemma 6.1, we deduce

Lemma 6.2. *If $\Sigma_{r,g} \neq 0$, then*

$$c_1(r,g)\ \Delta^{g/3}\ |\xi_1|^{3r+1-g}\ |\xi_2|^{-2} + c_2(r,g)\ \Delta^{r-g/6}\ |\xi_2|\ |\xi_1|^{-3r-2(1-g)} > 1 \tag{6.4}$$

where we may take

$$c_1(r,g) = \frac{1}{\sqrt{r}}\ 4^r \text{ and } c_2(r,g) = \frac{1}{\sqrt{r}}\ (2.252)^r,$$

for $r \geq 9$, and as follows, for $1 \leq r \leq 8$:

(r,g)	$c_1(r,g)$	$c_2(r,g)$	(r,g)	$c_1(r,g)$	$c_2(r,g)$
$(1,1)$	2.6	1.3	$(6,1)$	42.1	2.4
$(1,0)$	1.1	0.7	$(6,0)$	66.8	2.7
$(2,0)$	6.1	2.4	$(7,1)$	547.0	16.9
$(3,0)$	1.1	0.3	$(7,0)$	39.5	0.9
$(4,0)$	14.2	1.8	$(8,1)$	745.9	13.0
$(5,0)$	9.2	0.7	$(8,0)$	236.9	3.1

Proof. This is essentially Lemma 5 of Evertse [24]. Arguing as in the proof of that lemma, we write

$$|\Lambda_{r,g}| = \frac{1}{C_{r,g}} |\xi_1|^{3r+1-g} |\xi_2| \left| \left(\frac{\eta_2}{\xi_2} - 1 \right) A_{r,g}(z_1) + z_1^{2r+1-g} F_{r,g}(z_1) \right|.$$

Since $|1 - z_1| = 1$ and $|z_1| < 10^{-9}$, we may apply Lemma 6.1 and inequality (5.4) to find that $|\Lambda_{r,g}|$ is bounded above by

$$\frac{1}{C_{r,g}} |\xi_1|^{3r+1-g} |\xi_2| \left(\binom{2r-g}{r} \frac{1.012\sqrt{\Delta}}{|\xi_2|^3} + \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}} \left(\frac{3.001\sqrt{\Delta}}{|\xi_1|^3} \right)^{2r+1-g} \right).$$

Comparing this with (6.3), we obtain inequality (6.4) for any $c_1(r,g)$ and $c_2(r,g)$ that majorize

$$\frac{1.012}{C_{r,g}} \, 2^{-2g/3} \binom{2r}{r} \tag{6.5}$$

and

$$\frac{2^{g/3}}{C_{r,g}} (3.001)^{2r+1-g} \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}}, \tag{6.6}$$

respectively.

Substituting the values for $C_{r,g}$ from (6.2) into (6.5) and (6.6) leads to the desired conclusion for $1 \leq r \leq 8$. We may therefore suppose $r \geq 9$. Applying an explicit version of Stirling's formula (see e.g. Theorem (5.44) of [54]),

$$\frac{1}{2\sqrt{k}} \, 4^k \leq \binom{2k}{k} < \frac{1}{\sqrt{\pi k}} \, 4^k, \tag{6.7}$$

for $k \in \mathbb{N}$. It follows that

$$\frac{1.012}{C_{r,g}} \, 2^{-2g/3} \binom{2r}{r} < \frac{1.012}{\sqrt{\pi r}} \, 4^r < \frac{1}{\sqrt{r}} \, 4^r.$$

Similarly, we may show that

$$\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r} < \frac{\sqrt{3}}{2\pi r}, \tag{6.8}$$

for $r \in \mathbb{N}$ and $g \in \{0,1\}$. This follows from observing, if we set

$$\mathfrak{B}_r = \binom{r-2/3}{r}\binom{r-1/3}{r} = \frac{\theta_r}{r},$$

that

$$\mathfrak{B}_{r+1} = \left(\frac{r^2+r+2/9}{r^2+r} \right) \frac{\theta_r}{r+1},$$

whereby

$$\theta_r = \theta_1 \prod_{k=1}^{r-1} \frac{k^2+k+2/9}{k^2+k}.$$

Since $\theta_1 = 2/9$ and

$$\prod_{k=1}^{\infty} \frac{k^2+k+2/9}{k^2+k} = \frac{9}{2\,\Gamma(1/3)\Gamma(2/3)} = \frac{9\sqrt{3}}{4\pi},$$

we obtain (6.8) upon noting that

$$\binom{r-2/3}{r} > \binom{r+1/3}{r+1},$$

for $r \in \mathbb{N}$. Consequently,

$$\frac{2^{g/3}}{C_{r,g}}(3.001)^{2r+1-g}\frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}} < \frac{(3.001)\sqrt{3}}{\pi\sqrt{r}}\left(\frac{r+1}{2r+1}\right)\left(\frac{3.001}{2}\right)^{2r}.$$

Since $r \geq 9$, we conclude as desired. □

We will see in the next section that, if $\Sigma_{r,g} \neq 0$, this lemma provides a gap principle for our solutions (i.e. a nontrivial lower bound for $|\xi_2|$ in terms of $|\xi_1|$). For small values of r, we explicitly prove nonvanishing of $\Sigma_{r,g}$ in the following:

Lemma 6.3. $\Sigma_{r,g} \neq 0$ *for* $(r,g) = (1,1), (1,0), (2,0), (3,0), (4,0)$ *and* $(5,0)$.

Proof. This is Lemma 6 of Evertse [24] for $(r,g) = (1,1), (1,0), (2,0)$ and $(3,0)$. We mimic his proof in the remaining cases.

Let us begin by defining

$$\overline{A}_{r,g}(z) = \frac{3^{\alpha(r,g)}}{C_{r,g}} A_{r,g}(z)$$

and

$$\overline{B}_{r,g}(z) = \frac{3^{\alpha(r,g)}}{C_{r,g}} B_{r,g}(z),$$

where $\alpha(r,g)$ is the smallest integer such that $\overline{A}_{r,g}(z)$ and $\overline{B}_{r,g}(z)$ have integral coefficients (that $\alpha(r,g)$ exists and is bounded above by $[3r/2]$ is relatively straightforward to show; see Chudnovsky [16], Lemma 3.1).

From (6.1), we can find, for each $r \in \mathbb{N}$, a polynomial $K_r(z) \in \mathbb{Z}[z]$, satisfying

$$\overline{A}_{r,0}(z)^3 - (1-z)\overline{B}_{r,0}(z)^3 = z^{2r+1}K_r(z).$$

In fact, we have

$$\overline{A}_{3,0}(z) = 81 - 135z + 63z^2 - 7z^3,$$

$$\overline{B}_{3,0}(z) = 81 - 108z + 36z^2 - 2z^3,$$

$$\overline{A}_{4,0}(z) = 3402 - 7371z + 5265z^2 - 1365z^3 + 91z^4,$$

$$\overline{B}_{4,0}(z) = 3402 - 6237z + 3564z^2 - 660z^3 + 22z^4,$$

$$K_{4,0}(z) = 756756 - 1513512z + 972153z^2 - 215397z^3 + 10648z^4,$$

$$\overline{A}_{5,0}(z) = 6561 - 17496z + 16848z^2 - 7020z^3 + 1170z^4 - 52z^5,$$

$$\overline{B}_{5,0}(z) = 6561 - 15309z + 12474z^2 - 4158z^3 + 495z^4 - 11z^5$$

and

$$K_{5,0}(z) = 341172 - 852930z + 757836z^2 - 283824z^3 + 40408z^4 - 1331z^5.$$

Let us further define binary forms A_r^* and B_r^* via

$$A_r^*(x,y) = x^r \overline{A}_{r,0}(y/x)$$

and

$$B_r^*(x,y) = x^r \overline{B}_{r,0}(y/x).$$

Suppose that $\Sigma_{r,0} = 0$ for some $r \in \{4,5\}$. Setting $u = \xi_1^3$ and $v = \xi_1^3 - \eta_1^3$ (so that $z_1 = v/u$), we thus have

$$\frac{\eta_2^3}{\xi_2^3} = \frac{(u-v)\left(B_r^*(u,v)\right)^3}{u\left(A_r^*(u,v)\right)^3}.$$

Arguing as in the proof of Lemma 6 of Evertse [24], if $\mathfrak{a}_r$ is the integral ideal in $M = \mathbb{Q}(\sqrt{-\Delta})$ generated by $u\left(A_r^*(u,v)\right)^3$ and $(u-v)\left(B_r^*(u,v)\right)^3$, and $N(\mathfrak{a}_r)$ is its absolute norm, then

$$\frac{N(\mathfrak{a}_r)^{1/2}|v|^{-3r-1}|z_1|^r 3\sqrt{\Delta}}{|K_r(z_1)|} \geq 1.$$

Since $|z_1| = 3\sqrt{\Delta}\,|\xi_1|^{-3}$, we therefore have

$$|\xi_1|^{3r} \leq \frac{N(\mathfrak{a}_r)^{1/2}|v|^{-3r-1}\left(3\sqrt{\Delta}\right)^{r+1}}{|K_r(z_1)|}. \tag{6.9}$$

Following [24], we will seek an upper bound for $N(\mathfrak{a}_r)^{1/2}|v|^{-3r-1}$. Define M_1 to be a finite extension of M in which the ideal generated by u and v is principal, say generated by w. Set $u_1 = u/w, v_1 = v/w$ and let the extension of $\mathfrak{a}_r$ to M_1 be $\mathfrak{b}_r$. If we define

$$\mathfrak{r}_r = \left(A_r^*(u_1,v_1), B_r^*(u_1,v_1)\right),$$

then (see Evertse [24], formula (46))

$$\mathfrak{b}_r \supset w^{3r+1} B_r^*(0,1)^3 \mathfrak{r}_r^3. \tag{6.10}$$

Setting

$$F_4(x,y) = -188811x^3 + 203940x^2y - 49638xy^2 + 1738y^3$$

and

$$G_4(x,y) = -188811x^3 + 266877x^2y - 96369xy^2 + 7189y^3,$$

we may verify that

$$F_4(x,y)A_4^*(x,y) - G_4(x,y)B_4^*(x,y) = 296352x^3y^4$$

while

$$B_3^*(x,y)A_4^*(x,y) - A_3^*(x,y)B_4^*(x,y) = -28y^7.$$

These two identities imply that

$$\mathfrak{r}_4 \supset \left(296352u_1^3v_1^4, 28v_1^7\right) \supset 296352\,(v_1)^4.$$

Since $B_4^*(0,1) = 22$, it follows from (6.10) that

$$\mathfrak{b}_4 \supset \left(6519744^3 w^{13} v_1^{12}\right) \supset (6519744)^3 \, (v)^{13}. \tag{6.11}$$

Similarly, in the case $r = 5$, if we let

$$F_5(x,y) = -1228041x^4 + 1898127x^3y - 865854x^2y^2 + 119350xy^3 - 2739y^4$$

and

$$G_5(x,y) = -1228041x^4 + 2307474x^3y - 1362114x^2y^2 + 272870xy^3 - 12948y^4,$$

we find that

$$F_5(x,y)A_5^*(x,y) - G_5(x,y)B_5^*(x,y) = 312741x^4y^5$$

while

$$B_4^*(x,y)A_5^*(x,y) - A_4^*(x,y)B_5^*(x,y) = -143y^9.$$

We therefore have

$$\mathfrak{r}_5 \supset \left(312741u_1^4v_1^5, 143v_1^9\right) \supset 312741 \, (v_1)^5 \, ,$$

and so, since $B_5^*(0,1) = -11$, (6.10) implies that

$$\mathfrak{b}_5 \supset \left(3440151^3 w^{16} v_1^{15}\right) \supset (3440151)^3 \, (v)^{16}. \tag{6.12}$$

From (6.11) and (6.12), then, we have the inequalities

$$N(\mathfrak{a}_4)^{1/2}|v|^{-13} \leq (6519744)^3$$

and

$$N(\mathfrak{a}_5)^{1/2}|v|^{-16} \leq (3440151)^3 \, .$$

Noting that $|z_1| < 10^{-9}$ implies

$$|K_4(z_1)| > 756755 \text{ and } |K_5(z_1)| > 341171,$$

we may apply (6.9) to conclude that

$$|\xi_1| < 25.85 \; \Delta^{5/24}, \text{ if } r = 4$$

and

$$|\xi_1| < 13.47 \; \Delta^{1/5}, \text{ if } r = 5.$$

In each case, since we assume $\Delta \geq 72000$, this contradicts inequality (5.5), completing the proof of the lemma. □

For larger values of r, it is too time-consuming to provide case-by-case proofs of the nonvanishing of $\Sigma_{r,g}$. Instead, we utilize the following easy lemma.

Lemma 6.4. *If $r \in \mathbb{N}$ and $h \in \{0,1\}$, then at least one of*

$$\{\Sigma_{r,0}, \Sigma_{r+h,1}\}$$

is nonzero.

Proof. Consider the determinant

$$\begin{vmatrix} A_{r,0}(z_1) & A_{r+h,1}(z_1) & \eta_1/\xi_1 \\ A_{r,0}(z_1) & A_{r+h,1}(z_1) & \eta_1/\xi_1 \\ B_{r,0}(z_1) & B_{r+h,1}(z_1) & \eta_2/\xi_2 \end{vmatrix}.$$

Expanding along the first row, we find that

$$\begin{aligned} 0 &= A_{r,0}(z_1)\Sigma_{r+h,1} - A_{r+h,1}(z_1)\Sigma_{r,0} \\ &\quad + \frac{\eta_2}{\xi_2}\left(A_{r,0}(z_1)B_{r+h,1}(z_1) - A_{r+h,1}(z_1)B_{r,0}(z_1)\right). \end{aligned}$$

If both $\Sigma_{r,0}$ and $\Sigma_{r+h,1}$ vanish, then we must have

$$A_{r,0}(z_1)B_{r+h,1}(z_1) - A_{r+h,1}(z_1)B_{r,0}(z_1) = 0,$$

contradicting Lemma 3(iii) of Evertse [24] (since $z_1 \neq 0$). $\square$

7. Proof of Theorem 1.4 for $D_F \geq 24000$

We will use Lemmata 6.2, 6.3 and 6.4 to iterate our gap principle, showing that $|\xi_2|$ is arbitrarily large in relation to $|\xi_1|$. Specifically, we will prove that

$$|\xi_2| > (2.3\Delta)^{-r}|\xi_1|^{3r+2} \tag{7.1}$$

for all $r \geq 2$. Since $|\xi_1| > 0.24\ \Delta$ and $\Delta \geq 72000$, this contradicts any *a priori* upper bound for $|\xi_2|$.

First, for $1 \leq r \leq 5$, we apply Lemma 6.3 in conjunction with Lemma 6.2. From Lemma 5.2,

$$|\xi_2| > 0.987\, |\xi_1|^2$$

and so, since $c_1(1,1) = 2.6$, we obtain

$$c_1(1,1)\ \Delta^{1/3}|\xi_1|^3|\xi_2|^{-2} < 2.669\ \Delta^{1/3}|\xi_1|^{-1}.$$

From $|\xi_1| > 0.24\ \Delta$ and $\Delta \geq 72000$, it therefore follows that

$$c_1(1,1)\ \Delta^{1/3}|\xi_1|^3|\xi_2|^{-2} < 0.007$$

whence, since Lemma 6.3 implies the nonvanishing of $\Sigma_{1,1}$, we may apply Lemma 6.2 (and the fact that $c_2(1,1) = 1.3$) to conclude that

$$|\xi_2| > 0.763\ \Delta^{-5/6}|\xi_1|^3.$$

Arguing similarly for $(r,g) = (1,0), (2,0), (3,0), (4,0)$ and $(5,0)$, we have

$$|\xi_2| > c_3(r,g)\ \Delta^{g/6-r}|\xi_1|^{3r+2(1-g)}$$

where the values for $c_3(r,g)$ are as given in the following table:

(r,g)	$c_3(r,g)$	(r,g)	$c_3(r,g)$
$(1,1)$	0.763	$(3,0)$	3.333
$(1,0)$	0.302	$(4,0)$	0.555
$(2,0)$	0.388	$(5,0)$	1.428

This verifies (7.1) for $2 \leq r \leq 5$. To show that (7.1) holds for $r > 5$, we use induction on r. Suppose that (7.1) is true for some $r \geq 5$. Then

$$c_1(r+1,0)\ |\xi_1|^{3r+4}|\xi_2|^{-2} < \frac{4^{r+1}}{\sqrt{r+1}}\ (2.3\Delta)^{2r}|\xi_1|^{-3r}.$$

Inequality (5.5), $\Delta \geq 72000$ and $r \geq 5$ therefore allow us to conclude that

$$c_1(r+1,0)\ |\xi_1|^{3r+4}|\xi_2|^{-2} < 0.001.$$

If $\Sigma_{r+1,0} \neq 0$, then, from Lemma 6.2, we have

$$|\xi_2| > \frac{0.999}{c_2(r+1,0)}\ \Delta^{-r-1}\ |\xi_1|^{3r+5},$$

and, since $c_2(r+1,0) \leq (2.252)^{r+1}/\sqrt{r+1}$, it follows that

$$|\xi_2| > (2.3\Delta)^{-r-1}\ |\xi_1|^{3r+5},$$

as desired. If, however, $\Sigma_{r+1,0} = 0$, then Lemma 6.4 implies that both $\Sigma_{r+1,1}$ and $\Sigma_{r+2,1}$ are nonzero. Our induction hypothesis, (5.5), $\Delta \geq 72000$ and $r \geq 5$, imply

$$c_1(r+1,1)\ \Delta^{1/3}|\xi_1|^{3r+3}|\xi|^{-2} < \frac{4^{r+1}}{\sqrt{r+1}}(2.3)^{2r}\Delta^{2r+1/3}|\xi_1|^{-3r-1} < 0.001$$

and thus we may apply Lemma 6.2 to obtain

$$|\xi_2| > \frac{0.999}{c_2(r+1,1)}\ \Delta^{-r-5/6}\ |\xi_1|^{3r+3} > (2.252)^{-r}\ \Delta^{-r-5/6}\ |\xi_1|^{3r+3}. \tag{7.2}$$

It follows from (5.5) and (7.2) that

$$c_1(r+2,1)\ \Delta^{1/3}\ |\xi_1|^{3r+6}|\xi_2|^{-2} < c_1(r+2,1)\left(\frac{(2.252)^2}{(0.24)^3}\Delta^{\frac{2-r}{r}}\right)^r.$$

Since we have

$$c_1(r+2,1) = \begin{cases} 16.9, & \text{if } r=5; \\ 13.0, & \text{if } r=6; \\ \frac{4^{r+2}}{\sqrt{r+2}}, & \text{if } r \geq 7, \end{cases}$$

the inequality

$$c_1(r+2,1)\ \Delta^{1/3}\ |\xi_1|^{3r+6}|\xi_2|^{-2} < 0.301$$

(corresponding to $r = 5$) obtains from $\Delta \geq 72000$. A final application of Lemma 6.2 leads to

$$|\xi_2| > \frac{0.699}{c_2(r+2,1)}\ \Delta^{-r-11/6}\ |\xi_1|^{3r+6}.$$

Since $c_2(r+2,1) \leq (2.252)^{r+2}/\sqrt{r+2}$, $|\xi_1| > 0.24\Delta$, $\Delta \geq 72000$ and $r \geq 5$, we conclude that

$$|\xi_2| > (2.3\Delta)^{-r-1}\ |\xi_1|^{3r+5}.$$

This completes our induction and hence the proof of Theorem 1.4 for forms of discriminant exceeding 24000.

8. Finding representative forms of small discriminant

It remains to deal with those binary cubic forms F with $0 < D_F < 24000$. In fact, we will completely solve equation (1.2) for representatives of every equivalent class of (irreducible) binary cubic forms with $0 < D_F \leq 10^6$. The results of these computations are tabulated in Section 9. Our approach combines a method of Davenport [18] (as refined by Belabas and Cohen [8] and [9]) for finding such representatives, with recent techniques from the theory of linear forms in logarithms of algebraic numbers and computational Diophantine approximation for solving the resulting Thue equations.

There are two algorithms of which the author is aware for determining all classes of irreducible binary cubic forms with (positive) discriminant below a given bound; both may be readily extended to find all distinct real cubic fields of bounded discriminant. This follows from the existence of a one-to-one correspondence between such fields and *primitive* integral irreducible binary cubic forms (i.e. those whose coefficients a, b, c and d contain no common factor). The first of these methods is outlined in some detail in §30 of Delone and Fadeev [21]. It relies on the fact (established in Chapter II of [21]) that there is a discriminant preserving, bijective map between $GL_2(\mathbb{Z})$-equivalence classes of irreducible integral binary cubic forms and rings with unit elements contained in rings of integers of cubic fields (i.e. the conjugate sets of orders of such fields). Following Delone and Fadeev, one restricts attention to cubic forms of the shape $x^3 + bx^2y + cxy^2 + dy^3$, with $b \in \{0, 1\}$, via the following lemma (see Delone and Fadeev [21], §28 or Pethő [42], Theorem 1):

Lemma 8.1. *If $F(x, y)$ is a binary form with integral coefficients, $D_F > 0$ and $N_F \geq 1$, then there exist integers b, c and d with*

$$F(x, y) \sim x^3 + bx^2y + cxy^2 + dy^3, \tag{8.1}$$

$b \in \{0, 1\}$, $c < 0$ and

$$0 < d < \frac{2}{9}|c|\sqrt{3|c|}, \text{ if } b = 0,$$

or

$$\frac{9c - 2 - 2\sqrt{(1 - 3c)^3}}{27} < d < \frac{9c - 2 + 2\sqrt{(1 - 3c)^3}}{27}, \text{ if } b = 1.$$

A computational difficulty with this approach is, that the forms of type (8.1) one is led to consider, may be equivalent and hence one must distinguish between inequivalent forms of equal discriminant via either the method of inverse Tschirnhausen transformations (see §13 of [21]) or, for instance, by applying a result of Wolfskill [61].

We will instead utilize a second, rather simpler, algorithm for finding classes of cubic forms of bounded discriminant, based on the classical notion of reduction, and first applied by Davenport [18] (see also [19]). We say that a quadratic form $H(x, y) = Ax^2 + Bxy + Cy^2$ with real coefficients is *strongly reduced* if $0 \leq B \leq A \leq C$ and $C > 0$. A cubic form $F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ of positive discriminant will be called *strongly reduced* if its Hessian $H(x, y) = Ax^2+Bxy+Cy^2$

is strongly reduced in the above sense and, additionally,

$$
(8.2)\qquad
\begin{array}{ll}
(1) & a > 0. \\
(2) & \text{If } B = 0, \text{ then } b > 0. \\
(3) & \text{If } A = B, \text{ then } |b| < |3a - b|. \\
(4) & \text{If } A = C, \text{ then } a \leq |d| \text{ and } |b| < |c| \text{ whenever } a = |d|.
\end{array}
$$

Our notion of strong reduction coincides with that of reduction in [8] and [9]. The reason we introduce this more stringent notation is the following result (for which the reader is directed to [8]):

Proposition 8.2. *If $F(x,y)$ is an irreducible cubic form with $D_F > 0$, then $F(x,y)$ is equivalent to a unique strongly reduced form.*

Our strategy, then, will be to count strongly reduced forms of bounded discriminant. We will make use of the following lemma (Proposition 5.5 of [8] or [9]; a slight sharpening of Lemma 1 of [18]):

Lemma 8.3. *If $F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ is a strongly reduced cubic form with $0 < D_F \leq X$, then if $b < 0$, necessarily $c < 0$, and we have the following inequalities:*

$$1 \leq a \leq \frac{2}{3\sqrt{3}} X^{1/4}, \quad -X^{1/4} \leq b \leq \frac{2}{\sqrt{3}} X^{1/4},$$

$$-\frac{\sqrt{3}}{4} X^{1/2} \leq bc \leq X^{1/2}, \quad -\frac{4}{27} X^{1/2} \leq ad \leq \frac{\sqrt{3}}{36} X^{1/2},$$

$$\frac{-35 - 13\sqrt{13}}{216} X \leq ac^3, b^3 d \leq \frac{-35 + 13\sqrt{13}}{216} X$$

and

$$0 \leq c^2(bc - 9ad) \leq \frac{4}{3} X.$$

As noted in [8] and [9], using only the inequalities in the above lemma, the number of quadruples (a,b,c,d) we must check to find representatives of all cubic forms with $0 < D_F \leq X$ is of order $O(X)$. The following theorem of Davenport [18] (as sharpened by Shintani [47]) shows that this is relatively efficient.

Theorem 8.4. *Let $H_3(0,X)$ denote the number of equivalence classes of integral cubic forms F, with $0 < D_F \leq X$. Then, as $X \to \infty$, we have, for $\epsilon > 0$,*

$$H_3(0,X) = \frac{\pi^2}{72} X + C X^{5/6} + O(X^{2/3+\epsilon}) \sim 0.137X,$$

where C is constant.

Note that the seeming discrepancy between this result and that stated in [18] derives from a missing factor of 3 in the statement of the main theorem of [18], together with the fact that the estimate in the latter is for classes of *properly equivalent*, rather than *equivalent*, forms; i.e. for $SL_2(\mathbb{Z})$ instead of $GL_2(\mathbb{Z})$ equivalence classes.

Applying Lemma 8.3 with $X = 10^6$, we may thus assume that

$$1 \leq a \leq 12 \text{ and } -31 \leq b \leq 36.$$

From the inequalities for c and d in Lemma 8.3, together with the more precise estimates

$$0 < D_F = 18abcd + b^2c^2 - 27a^2d^2 - 4ac^3 - 4b^3d \leq 10^6$$

and (8.2), we find, upon checking for reducibility, that there are precisely 89595 classes of irreducible binary cubic integral forms F with $0 < D_F \leq 10^6$. For each of these, we must solve equation (1.2).

9. Results of our computations

Solving Thue equations of low degree has, in recent years, become a relatively routine matter. The standard approach to this problem tranforms a given Thue equation into an equation for units in a corresponding number field and then derives an upper bound upon solutions to (1.1) from lower bounds for linear forms in complex logarithms and explicit information about fundamental (or, perhaps, only independent) units in the field. The Lenstra-Lenstra-Lovasz lattice-basis reduction algorithm can then be used to reduce these bounds to a reasonable size. The best reference for our purposes is [60], while newer innovations are outlined in [51]; we direct the reader to either of these sources for details of these methods. The bottleneck, from a computational viewpoint, in any of these approaches, is the computation of the related fundamental (or independent) units. However, for the cubic fields we are concerned with, this does not present major difficulties. Using code written in C and utilizing Pari GP, Version 1.39 to compute our fundamental units and perform our lattice-basis reduction, we solved each of the 89595 equations of the form (1.2) corresponding to irreducible binary cubic forms F with $0 < D_F \leq 10^6$. We double-checked our results using Kant V4 (Version 2.0, Jan. 1999) on a DEC Alpha 21164A, running at 433MHz and Kant V4 (Version 2.1, May 1999) on a Sun Ultra 10, running at 333 MHz. With the latter, more recent release, we ran into problems only with the form

$$F(x, y) = 6x^3 + 8x^2y - 29xy^2 - 7y^3$$

of discriminant $D_F = 781260$. In this case, memory constraints made it difficult to use Kant to solve equation (1.2) (though, via Pari, we had no such problems). The problem here is the size of a system of fundamental units ϵ_1 and ϵ_2 in $\mathbb{Q}(\theta)$, where

$$\theta^3 + 8\theta^2 - 174\theta - 252 = 0.$$

These are given, in terms of the integral basis

$$\left\{1, \theta, \frac{1}{6}\left(\theta^2 + 2\theta\right)\right\},$$

by

$$\epsilon_1 = -39 - 30\theta + 16\frac{1}{6}\left(\theta^2 + 2\theta\right)$$

and

$$\epsilon_2 = a_1 + a_2\theta + \frac{a_3}{6}\left(\theta^2 + 2\theta\right),$$

where

$$a_1 = -115017342781184445090262418849486253520 33,$$

$$a_2 = 29042832482768449552806772167202417994 9$$

and

$$a_3 = 37686804032541135405916534534550829962 4.$$

In conjunction with sharp lower bounds for linear forms in logarithms of algebraic numbers, due to Baker and Wustholz [5]; however, this still leads to a lattice basis reduction problem of reasonable size (and to the conclusion that the equation $6x^3 + 8x^2y - 29xy^2 - 7y^3 = 1$ has no integral solutions).

In each of our 89595 cases, we have $N_F \leq 9$ and, in fact, we found nothing to contradict Conjecture 1.7. If we write $R(X,N)$ for the number of irreducible integral cubic forms F with discriminant $0 < D_F \leq X$, for which equation (1.2) has exactly N distinct solutions in integers, then we have

N_F	$R(24000, N_F)$	$R(10^6, N_F)$
0	323	49687
1	603	32992
2	326	6088
3	109	638
4	29	146
5	13	39
6	4	4
9	1	1

We note that there are precisely seven classes of forms with $0 < D_F \leq 10^6$ and $N_F = 5$ which are inequivalent to all forms in the families

$$F_m(x,y) = x^3 - (m+1)x^2y + mxy^2 + y^3$$

and

$$G_n(x,y) = x^3 - n^2x^2y + y^3.$$

These are given by $F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, where

a	b	c	d	D_F	Solutions to $F(x,y) = 1$
1	1	-3	-1	148	$(1,0),(0,-1),(-2,1),(3,2),(-14,45)$
1	0	-5	-1	473	$(1,0),(0,-1),(-2,1),(1,-5),(7,3)$
1	0	-7	-1	1345	$(1,0),(0,-1),(1,-7),(18,-7),(-19,-7)$
1	11	-1	-12	62501	$(1,0),(1,-1),(-1,-1),(11,-1),(-179,-172)$
1	9	-12	-21	108729	$(1,0),(1,-1),(-2,-1),(10,-1),(4651,2294)$
1	21	-2	-21	783689	$(1,0),(-1,1),(-1,-1),(356,-365),(442,-21)$
1	21	-1	-22	810661	$(1,0),(1,-1),(-1,-1),(21,-1),(373,-364)$

Five of these classes were observed by Pethő [42] (the discriminant of the form with $D_F = 62501$ is given incorrectly in [42]), while that with $D_F = 108729$ was found by Lippok [32]. The class with $D_F = 783689$ appears to be new. Note that the forms with $D_F = 148, 473$ and 783689 are all special cases of the parametrized family given by

$$H_n(x,y) = x^3 + nx^2y - 2xy^2 - ny^3,$$

with $n = 2, 3$ and 21, respectively ($n = 0$ and 1 correspond to forms of discriminant 32 and 49; the first of these represents the unique reducible class of forms with $N_F =$

3 while the second has $N_F = 9$). For arbitrary $n \in \mathbb{Z}$, the equation $H_n(x,y) = 1$ has the solutions $(x,y) = (1,0), (-1,1), (-1,-1)$ and $(n^2+1, -n)$ and discriminant

$$D_{H_n} = 4n^4 + 13n^2 + 32.$$

It is worth observing that this family of forms is apparently inequivalent to

$$F(x,y) = (x - a_1 y)(x - a_2 y)(x - a_3 y) \pm y^3$$

for a_1, a_2 and a_3 integral, provided $n \geq 2$ (i.e. $H_n(x,y)$ is not a split family).

10. Concluding remarks

The arguments of Sections 6 and 7, together with a very slight refinement of Lemma 5.2, may be used to show, if $c, \delta > 0$, that there is at most a single integral solution (x,y) to (1.2), related to a pair of resolvent forms (ξ, η), for which

$$|\xi(x,y)| \geq c\, \Delta^{2/3+\delta},$$

provided Δ is suitably large (in terms of c and δ). To sharpen Theorem 1.4 by proving that $N_F \leq 7$ for large D_F, one would, in all likelihood, need to significantly strengthen Lemma 5.2.

As a final remark, we mention that Chudnovsky and Chudnovsky [17] claim to have proven that $N_F \leq 9$ for all cubic forms with sufficiently large discriminant. While we believe this to be true (indeed refer to the stronger Conjecture 1.7), there is no proof of this assertion given in [17] and hence this author has no way of determining its validity.

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