Recent advances in the MMP after Shokurov I

Introduction to the MMP

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(Smooth Projective) Curves

- Curves C come in three types:
  - $C \simeq \mathbb{P}^1$
  - C is a plane cubic
  - C has genus ≥ 2

- The goal of higher dimensional geometry is to obtain a classification of varieties, modeled after the classification of curves.

Topology of Curves

- their fundamental groups come in three types:

- $C \simeq \mathbb{P}^1$ — π_1 is trivial
- C is a plane cubic — π_1 is a free abelian group
- C has genus ≥ 2 — π_1 is non-commutative

- The goal of higher dimensional geometry is to obtain a classification of varieties, modeled after the classification of curves.

Arithmetic of Curves

- ...rational points on them come in three types:

- $C \simeq \mathbb{P}^1$ — non-finitely generated
- C is a plane cubic — finitely generated
- C has genus ≥ 2 — finite

- The goal of higher dimensional geometry is to obtain a classification of varieties, modeled after the classification of curves.

Differential Geometry of Curves

- ...their curvatures come in three types:

- $C \simeq \mathbb{P}^1$ positively curved
- C is a plane cubic flat
- C has genus ≥ 2 negatively curved

- The goal of higher dimensional geometry is to obtain a classification of varieties, modeled after the classification of curves.

Curves (continued)

- Curves C come in three types (yet again):
- K_C is negative.
- K_C is trivial.
- K_C is positive.

Hypersurfaces

- Hypersurfaces $X \subset \mathbb{P}^n$ also come in three types:
- $\deg X < n+1$ — K_X is negative.
- $\deg X = n+1$ — K_X is trivial.
- $\deg X > n+1$ — K_X is positive.

Curves (take three)

- Curves C actually come in two types.

Kodaira dimension

- K_C is negative. $\kappa(C) = -\infty$

  $|mK_C| = \emptyset$ for all $m > 0$

- K_C is non-negative. $\kappa(C) \geq 0$

  $|mK_C| \neq \emptyset$ for some $m > 0$

Surfaces

- are more complicated...
- Example 1: $S = \mathbb{P}^1 \times C$ where C is a curve of genus at least two. K_S is neither negative nor positive.
- Example 2: T is a surface with K_T ample. Let S be the blowing up of T at a (smooth) point. K_S is no longer ample, not even nef: $K_S \cdot E = -1$, where E is the exceptional curve.

Surfaces (continued)

- Surfaces S come in three types:
- S is birational to $\mathbb{P}^2$.
- S is birational to a ruled surface.
- S is birational to T with K_T non-negative.

Surfaces (continued)

- A surface S is birational to T and $\exists\, \sigma : T \to Z$ such that K_Z is non-negative, and either
- $T = \mathbb{P}^2$ and Z is 0-dimensional, or
- $\sigma^{-1}(z) \simeq \mathbb{P}^1$, and Z is 1-dimensional, or
- $T = Z$, i.e., Z is 2-dimensional.

Surfaces (continued)

- A surface S is birational to T and $\exists\, \sigma\colon T \to Z$ such that
- K_Z is non-negative, and
- $K_{\sigma^{-1}(z)}$ is negative.

Wishful Thinking

- We hope that any variety admits a birational model X with a fibration, $\sigma\colon X \longrightarrow Z$ such that
  - K_Z is non-negative, and
  - $K_{\sigma^{-1}(z)}$ is negative.

- N.B.: The two extreme cases are also included:
  - $X = Z$,
  - K_X is negative and Z is a point.

The algorithm (surface case)

- Start with any birational model X.
- Desingularize X.
- If K_X is nef then **STOP**
- Otherwise there is a curve C, such that $K_X \cdot C < 0$. Our aim is to remove this curve or reduce the question to a lower dimensional one.
- There are two cases...

The algorithm (surface case)

- There are two cases:
  - If C moves in a positive dimensional family, then that family covers X, and we are dealing with a birationally ruled surface: **STOP**.
  - If C does not move, then it's a (-1)-curve, so by Castelnuovo's Theorem it can be contracted. If after this K_X becomes nef then **STOP**, otherwise repeat the process.
- Notice that in both cases we ended up with a map that contracted the curve C, and those (and only those) curves that must be contracted with C.

The algorithm (general case)

- Start with any birational model X.
- Desingularize X.
- If K_X is nef then **STOP**
- Otherwise there is a curve C, such that $K_X \cdot C < 0$. Our aim is to remove this curve or reduce the question to a lower dimensional one.
- By the Cone Theorem, there is an extremal contraction, $\pi\colon X \longrightarrow Y$, of relative Picard number one such that for a curve C', $\pi(C')$ is a point iff C' is homologous to C.

Analyzing $\pi : X \to Y$

- If the fibres of π have dimension at least one, then we have a Mori fibre space, that is, $-K_X$ is π-ample, π has connected fibres and relative Picard number one. We have reduced the question to a lower dimensional one: **STOP**.
- If π is birational and the locus contracted by π is a divisor, then even though Y might be singular, it will at least be $\mathbb{Q}$-factorial (for every Weil divisor D, some multiple is Cartier).
  If K_Y is nef, then **STOP**.
  Otherwise replace X by Y and keep going.

π is birational (continued)

- If the locus contracted by π is not a divisor, that is, π is small, then Y is not $\mathbb{Q}$-factorial:
- If K_Y were $\mathbb{Q}$-Cartier, then $\pi^* K_Y$ would (at least numerically) make sense. Since π is small, this would imply that K_X and $\pi^* K_Y$ are numerically equivalent. However, that is not possible as

$$K_X \cdot C < 0 \quad \text{by assumption, and}$$
$$\pi^* K_Y \cdot C = 0 \quad \text{because } C \text{ is contracted by } \pi.$$

- This is a brand **new case** that we haven't encountered in lower dimensions.

WE ARE STUCK!

Are we really stuck?

- Instead of contracting C, we can try to replace X by another birational model X^+, $X \dashrightarrow X^+$, such that the induced map $\pi^+ \colon X^+ \longrightarrow Y$ is a morphism and K_{X^+} is π^+-ample.

  This operation is called a flip.
- However, even supposing we can perform a flip, how do know that this process terminates? It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?

Two Conjectures

To finish the proof of the existence of the MMP, we need to prove the following two conjectures:

Conjecture. *(Existence of Flips) Let* $\pi\colon X \longrightarrow Y$ *be a small extremal contraction. Then the flip of* π *exists.*

Conjecture. *(Termination of Flips) There is no infinite sequence of flips.*

History

- The existence of contractions was established in the 80's (the Cone Theorem), and is due to Kawamata, Kollár, Mori, Reid and Shokurov. The proof uses Kawamata's X-method (cf. next lecture).
  It works in all dimensions.
- The existence of flips for threefolds was established by Mori in the late 80's. Subsequently, Shokurov proved the existence of threefold log flips, in the early 90's. Both proofs rely heavily on classification.
- Termination is known for threefolds and for terminal fourfolds.

General Set up

Let $\pi\colon X \longrightarrow Y$ be a small contraction, of relative Picard number one where $-K_X$ is π-ample. We want to construct the flip.

That is, we want a diagram

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X^+ \\ & \searrow^{\pi} & & {}^{\pi^+}\swarrow & \\ & & Y & & \end{array}$$

where K_{X^+} is π^+-ample.

Finite Generation

- Let $\pi\colon X \longrightarrow Y$ be a small contraction, of relative Picard number one where $-K_X$ is π-ample. We (still) want to construct the flip.
- The claim is that the flip exists iff the ring

$$R = R(X, K_X) = \bigoplus_n H^0(X, \pi_* \mathcal{O}_X(nK_X)),$$

is finitely generated.

- In fact $X^+ = \operatorname{Proj}_Y R$.

What's Next? Abundance

- Starting with an arbitrary X, the MMP provides a birational model, which is a Fano fibre space over a minimal variety:
  $X \sim X' \to Z$, where K_Z is nef.

  Conjecture. *(Abundance) If K_Z is nef, then $|mK_Z|$ is base point free for some $m > 0$.*
- One obtains a pluricanonical map, $Z \to \mathbb{P}^N$, a new tool to study Z.