Some Problems Concerning Forbidden Configurations

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Abstract

For an $m \times n$ (0,1)-matrix with no repeated columns and no submatrix which is a row and column permutation of any matrix in some set $\mathfrak{F}$, we attempt to determine the maximum number of columns n. We consider for example $\mathfrak{F}$ as the set of square (0,1)-matrices with row and column sums 2 and order at least 4 and obtain the asymptotically exact bound: n is $O(m^2)$. We also consider $\mathfrak{F}$ consisting of a single $2 \times 2k$ matrix of k copies of the identity of order 2 and get the asymptotically exact bound: n is $O(m)$. These examples are improvements on the general bounds determined by results of Sauer, Perles and Shelah and others.

Section 1 Introduction.

A number of combinatorial objects can be encoded as (0,1)-matrices often defined via forbidden substructures. We will use the term configuration (the combinatorial equivalent of a submatrix) as follows. For a matrix B, we say a matrix A has no configuration B if A has no submatrix which is a row and column permutation of B. Let K_k be a $k\times 2^k$ (0,1)-matrix of all possible (0,1)-columns on k rows. Define a matrix to be simple if it is a (0,1)-matrix and has no repeated columns. The following result is a basic extremal result for forbidden configurations and gives the form of the results we seek in this paper.

Theorem 1.1 (Sauer [7], Perles and Shelah [8]) Let A be an $m\times n$ (0,1)-matrix with no configuration K_k. Then

$$n \le \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0} \tag{1.1}$$

and (1.1) is best possible. □

Now for any $k\times \ell$ matrix F we see that $K_{k+\lceil \log_2 \ell \rceil}$ contains F as a configuration, the extra $\lceil \log_2 \ell \rceil$ coming from possible repeated columns in F. This yields that if F is forbidden as a configuration as in Theorem 1.1 that n is $O(m^{k+\lceil \log_2 \ell \rceil -1})$ but we can do better. Let $t\cdot A$ denote the matrix consisting of t copies of A.

Theorem 1.2. Let A be an $m\times n$ (0,1)-matrix with no configuration $t\cdot K_k$. Then n is $O(m^k)$ or more precisely (with $t>1$)

$$n \le \binom{m}{k} + \binom{m}{k-1} + \dots + \binom{m}{0} + \frac{t-2}{k+1}\binom{m}{k}, \tag{1.2}$$

and this is asymptotically exact namely we can find such an A with

$$n = \binom{m}{k} + \binom{m}{k-1} + \dots + \binom{m}{0} + (1-o(1))\frac{t-2}{k+1}\binom{m}{k}. \tag{1.3}$$

Proof. The bound $O(m^k)$ is due to Füredi [6]. An expression for the exact bound is in [4] but it reduces to the maximum p so that there is an $m\times p$ simple matrix with no $k\times t$ submatrix of 1's and so we apply [Lemma 3.1, 1]. □

Now when forbidding F, we get that n is $O(m^k)$. Often it is of interest to study families of forbidden configurations. Let C_k denote a $k\times k$ (0,1)-matrix of the vertex-edge incidence matrix of a cycle of length k. A $m\times n$ (0,1)-matrix A is balanced (respectively totally balanced) if it has no configuration C_k for $k\ge 3$ and

k odd (resp. for all $k \geq 3$). Now if A is simple then

$$n \leq \binom{m}{2} + \binom{m}{1} + \binom{m}{0} \tag{1.4}$$

the bound simply arising from forbidding C_3 alone. That the bound is best possible is somewhat remarkable [2]. Partly motivated by the fact that the row and column sums of C_k are all two, Section 2 studies the bounds arising from forbidding square submatrices of constant row and column sums k. The most intriguing result is a bound of $O(m^2)$ from forbidding the 4×4 (0,1)-matrix which is a direct sum of C_2 with itself where the bound $O(m^4)$ might be expected via Theorem 1.2. But such is the preliminary state of our knowledge that asymptotically exact bounds are not known for C_5 or say C_2 direct sum with C_3.

Section 3 was motivated by (but does not solve) the problem of forbidding $2 \cdot C_3$. In particular, the construction of an $m \times \Omega(m^k)$ simple matrix for Theorem 1.2 relies on avoiding the $k \times t$ matrix of 1's, $t > 1$.

Problem 1.3. <u>Let</u> F <u>be a</u> $k \times \ell$ (0,1)-<u>matrix with at most one column of</u> 0's <u>and at most one column of</u> 1's. <u>If</u> A <u>is an</u> $m \times n$ <u>simple matrix with no configuration</u> F, <u>then is it true that</u> n <u>is</u> $O(m^{k-1})$? □

A result to support this is given as well as two other results when the general bounds discussed above are not accurate. As the answers about forbidden configurations become more detailed, we are given the ability to ask more detailed questions.

Section 2. Forbidden submatrices of constant line sums

We wish to consider the bounds resulting from forbidding a structured family of configurations. Let

$$\mathfrak{F}_{k,\ell} = \{A \mid A \text{ is a (0,1)-matrix of order } t \geq \ell \text{ and all row and column sums are } k\}. \tag{2.1}$$

Obviously $k \leq \ell$ and $\mathfrak{F}_{k,\ell+1} \subseteq \mathfrak{F}_{k,\ell}$. Let J_m denote the matrix of all 1's of order m and I_m denote the identity matrix of order m. Now $J_k \in \mathfrak{F}_{k,k}$ and $J_{k+1} - I_{k+1} \in \mathfrak{F}_{k,k+1}$. To obtain bounds for J_k use Theorem 1.2 to get asymptotically exact results. For $J_{k+1} - I_{k+1}$ consider the following.

Proposition 2.1 (Theorem 3.4[3]) <u>Let</u> A <u>be an</u> $m \times n$ <u>simple matrix and let</u> F <u>be a</u> $t \times s$ <u>simple matrix</u>(e.g. $J_{k+1} - I_{k+1}$). <u>Assume</u> A <u>has no configuration</u> F. <u>Then</u>

$$n \leq \binom{m}{t-1} + \binom{m}{t-2} + \dots + \binom{m}{0}. \tag{2.2}$$

Proof. This follows from Theorem 1.1 since F is a configuration in K_t. □

Note that for $k < \ell$, there is always an $F \in \mathcal{F}_{k,\ell}$ of order ℓ which is simple. To obtain constructions of matrices with no configuration in $\mathcal{F}_{k,\ell}$ note the following.

Proposition 2.2. Let $F \in \mathcal{F}_{k,\ell}$. For $\ell > k$, any row and column permutation of F contains the $(k+1)\times 1$ and $(\ell-k+1)\times 1$ submatrices

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{2.3}$$

where the 1's below the first entry of the first column number k, and the 0's below the first entry of the second column number $\ell-k$.

Proof. For the first $(k+1)\times 1$ column note that each row of F contains a zero ($\ell > k$) and so choose a column in F with a 0 in row 1 which then has k 1's below. For the second $(\ell-k+1)\times 1$ column choose a column in F with a 1 in row 1 which then has $\ell-k$ 0's below. □

In certain cases other columns can be shown to be submatrices but they are of no use in the following construction.

Proposition 2.3 ([1]). Let α be a $p\times 1$ (0,1)-column. Then there is an $m\times n$ simple matrix A with no submatrix α with

$$n = \binom{m}{p-1} + \binom{m}{p-2} + \dots + \binom{m}{0} \quad \square \tag{2.4}$$

Combine to obtain the following.

Theorem 2.4. Let A be an $m\times n$ simple matrix and $k < \ell$ given. Assuming A has no configuration $F \in \mathcal{F}_{k,\ell}$. Then

$$n \leq \binom{m}{\ell-1} + \binom{m}{\ell-2} + \dots + \binom{m}{0} \tag{2.5}$$

i.e. n is $O(m^{\ell-1})$ and there exists such an A with, for $p = \max\{k+1, \ell-k+1\}$,

$$n \geq \binom{m}{p-1} + \binom{m}{p-2} + \dots + \binom{m}{0} \tag{2.6}$$

i.e. n is $\Omega\ (m^{\max(\ell-k,k)})$

Proof. Proposition 2.1 and the note following it give (2.5). Propositions 2.2 and 2.3 yield (2.6). □

Note that for $\ell = k + 1$ or $k = 1$ we have exact solutions (the case $k = 1$ would follow from Füredi and Quinn [5]). Perhaps the constructions are best possible. We now consider some cases not solved by Theorem 2.4, in particular $k = \ell = 2$, $k = \ell$ in general and $k = 2$, $\ell = 4$ for which an asymptotic result is obtained.

Theorem 2.5. Let A be an $m\times n$ simple matrix with no configuration in $\mathfrak{F}_{2,2}$. Then $n\leq 2m$ and this is best possible.

Proof. Delete from A th columns with zero or one 1 (up to m+1). Consider the remaining columns of A as edges of a hypergraph H. We get a forest structure (a minimal cycle in H will yield a matrix $F \in \mathfrak{F}_{2,2}$ of the same order as the cycle) which can be built inductively by adding edges that overlap the previously constructed tree in zero or one vertex. Thus it has the maximum number of edges if they are all of size 2 in which case H has at most m-1 edges (achievable by the $m\times(m-1)$ vertex-edge incidence matrix of a tree on m vertices). Now $(m+1) + (m-1)$ yield the bound. □

What about $\mathfrak{F}_{k,k}$ for $k > 2$ where Theorem 2.4 is not helpful. Perhaps the matrix

$$\left[\begin{array}{c|c} K_m^0 K_m^{k-1} & \begin{matrix} 11\dots1 \\ K_{m-1}^{k-1} \end{matrix} \end{array}\right] \tag{2.7}$$

is extremal in that case, where K_q^p denotes the $p\times\binom{p}{q}$ simple matrix of all columns of q 1's. We now present a forbidden configuration result for a certain $F\in \mathfrak{F}_{2,4}$ given in (2.8) that gives the asymptotically correct bound for $\mathfrak{F}_{2,4}$ of $O(m^2)$; the construction of $\Omega(m^2)$ being given in Theorem 2.4.

Theorem 2.6. Let A be an $m\times n$ simple matrix with no configuration

$$\begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \end{bmatrix}. \tag{2.8}$$

Then n is $O(m^2)$ and this is best possible asymptotically.

Proof. We will use induction on m and Lemma 2.7 which follows. Decompose A by permuting columns to obtain

$$A=\begin{bmatrix} 11...100...0 \\ B_1B_2\ B_2B_3 \end{bmatrix} \tag{2.9}$$

by deleting the first row of A and identifying B_2 as the repeated columns on m-1 rows. Now B_2 is simple and has no configurations F_1 , F_2 as given in (2.10) below, since either forces (2.8) in A. By Lemma 2.7, B_2 has at most 9(m-1) columns. Now $[B_1B_2B_3]$ is also simple and has no configuration (2.8) and so we may apply induction to get that $[B_1B_2B_3]$ has at most $k(m-1)^2$ columns for some $k > 4^1/_2$. But then A has at most $k(m-1)^2 + 9(m-1) < k(m^2)$ columns, establishing the result.

Noting that the configuration (2.8) is in $\mathfrak{F}_{2,4}$, then (2.6) in Theorem 2.4 establishes that the bound is asymptotically exact. □

Lemma 2.7 Let A be an m×n simple matrix with no configurations

$$F_1=\begin{bmatrix} 10 \\ 10 \\ 01 \\ 01 \end{bmatrix},\ F_2=\begin{bmatrix} 1100 \\ 0011 \\ 0011 \end{bmatrix} \tag{2.10}$$

Then $n \le 9m$.

Proof We ignore in A the column of 0's and the column of 1's and for columns of column sum k ($1 \le k \le m-1$) we may ignore up to three more (two of which are chosen in a special way below). A total of 3m-1 might be ignored in this way.

Let A_k denote the columns of column sum k. We will assume A_k has at least 4 columns. In view of F_1 being forbidden we get either that all columns in A_k have k-1 1's in a (k-1)-subset of rows (which we call type 1 pattern) or all columns in A_k have m-k-1 0's in an (m-k-1)-subset of the rows (which we call a type 2 pattern) and is the (0,1)-complement of type 1. For A_k having 3 or more columns this classification is unique.

Now we will restrict attention to columns of column sums yielding type 1 patterns. The bound for type 2 patterns will be the same since it is just the (0,1)-complement. Let column sums $s_1 \le s_2 \le ... \le s_t$ yield type 1 patterns. For each s_i. Let B_i denote the (k-1)-subset of the rows which has all 1's in columns of column sum s_i and let S_i be those rows containing a 1 in exactly one column of column sum s_i. Note that $|S_i| \ge 4$. Now for $s_i < s_j$, we deduce that $B_i \subseteq B_j$. For $s_i = 1, B_i = \phi$ so this is trivial. For $s_i > 1$, if $B_i \backslash B_j = \phi$ then we get the configuration F_2 with the first row from $B_i \backslash B_j$ and the second two rows from $B_j \backslash B_i$ and the first two columns from those of column sum s_i (we might need four to choose from) and the second two columns from those of column sum s_j (we might need three to choose from).

Reorder the rows so that for each i, $B_i = \{1,2,\ldots,s_i-1\}$. Now we choose two of the three columns of column sum s_i to delete to ensure that when S_i' denotes those rows containing a 1 in exactly one of the remaining columns of columns sum s_i, then $s_i, s_{i+1} \notin S_i'$. We can show that for each $p = 5,6,\ldots,m$, there are at most 3 S_i''s containing it. Otherwise let $p \in S_i', S_j', S_k', S_\ell'$, with $i < j < k < \ell$. Then $s_j + 2 \leq s_\ell \leq p$. But now we get the configuration F_2 in rows s_j, s_j+1, p and in the two columns putting p in S_i' and S_j' and in the two columns of column sum s_ℓ not putting $p \in S_\ell'$. This shows

$$\sum_{i=1}^{t} |S_i'| \leq 3(m-4) \tag{2.11}$$

We get a similar bound for type 2 patterns and have not considered up to 3m-1 other columns. Thus the bound is

$$n \leq 3(m-4) + 3(m-4) + 3m-1 \leq 9m \qquad \square \tag{2.12}$$

Note that same argument will give n is O(m) when forbidding F_1 and $t{\cdot}F_2$. The lemma is surely not best possible but note that the best construction for Theorem 2.6 has B_2 being an identity matrix of order m-1.

Section 3. Beating the general bounds.

Let F be a $k \times \ell$ (0,1)-matrix and let A be an $m \times n$ simple matrix with no configuration F. Now from Section 1 we have seen that n is $O(m^k)$ and if F is simple we get n is $O(m^{k-1})$ from Proposition 2.1. Can we do better? This section offers three examples.

The following result provides some evidence for Problem 1.3 but it is premature to make a conjecture. Let I_2 denote the 2×2 identity matrix.

Theorem 3.1. <u>Let</u> A <u>be</u> <u>an</u> $m \times n$ simple <u>matrix</u> <u>with</u> <u>no</u> $2 \times 2k$ <u>configuration</u> $k{\cdot}I_2$. <u>Then</u> n <u>is</u> O(m).

<u>Proof</u>. We will actually show $n \leq (2k-2)m+1$ for $k \geq 2$, and use induction on m. It is

true for $m = 1,2$. For $S \subseteq \{1,2,\ldots,m\}$, let $A|_S$ be the submatrix of A of those rows indexed by S. Now for $i < j$ assume that the number of columns in $A|_{\{i,j\}}$ that are either

$\begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$ is at most $2k-2$. Then we can delete row i of A and those columns α of A which have $\alpha|_{\{i,j\}} = \begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$, to obtain a simple matrix B on $m-1$ rows and at least $n-(2k-2)$ columns and no configuration $k\cdot I_2$. Then induction yields $n \leq (2k-2)m+1$. Thus we may assume for $i < j$, that the number of columns $\begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$ in $A|_{\{i,j\}}$ is at least $2k-1$.

Form a tournament T whose vertices are the rows of A as follows. For $i < j$, the forbidden configuration $k\cdot I_2$ forces that either $A|_{\{i,j\}}$ has at most $k-1$ $\begin{bmatrix}1\\0\end{bmatrix}$'s and so at least k $\begin{bmatrix}0\\1\end{bmatrix}$'s in which case we define in T that $i \to j$ or $A|_{\{i,j\}}$ has at most $k-1$ $\begin{bmatrix}0\\1\end{bmatrix}$'s and so at least k $\begin{bmatrix}1\\0\end{bmatrix}$'s in which case we define in T that $j \to i$. Now T is a tournament and so has a hamiltonian path. Hence we may reorder the vertices so that $j \to j+1$ for each $1 \leq j \leq m-1$.

Now there are exactly $m+1$ columns with no 'fall', that is no $\begin{bmatrix}1\\0\end{bmatrix}$ in a consecutive pair of rows. Then if A has $m+1+(k-1)(m-1)+1$ columns then A has $(k-1)(m-1)+1$ $\begin{bmatrix}1\\0\end{bmatrix}$'s in consecutive pairs of rows and hence, by the pigeonhole principle, there are at least k $\begin{bmatrix}1\\0\end{bmatrix}$'s in some consecutive pair of rows $j,j+1$. This contradicts $j \to j+1$. Thus in this case $n \leq km+1$. This completes the proof. $\square$

It is likely the construction in (Theorem 3.6,[1]) for an $m\times n$ simple matrix with no submatrix $(k-1)\cdot\begin{bmatrix}1\\0\end{bmatrix}$, yields the correct bound. A similar preliminary analysis can be done for the $2\times(2k+2)$ forbidden configuration of $k\cdot I_2$ with $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ appended. But the outcome is unclear.

The following is an easy case where the general bounds do not apply.

Proposition 3.2. <u>Let</u> α <u>be a</u> $k\times 1$ <u>(0,1)-column and let</u> A <u>be an</u> $m\times n$ <u>simple matrix with no configuration</u> α. <u>Assume</u> α <u>has</u> p 1's <u>and</u> q 0's. <u>Then</u>

$$n \le \binom{m}{p-1}+\binom{m}{p-2}+\dots+\binom{m}{0}+\binom{m}{q-1}+\binom{m}{q-2}+\dots+\binom{m}{0} \quad \square \tag{3.2}$$

The following is a case where the 3×2 matrix F has no repeated columns yet the bound turns out to be $O(m)$ improving on $O(m^2)$. How do we predict such results in advance?

Theorem 3.3. Let A be an $m\times n$ simple matrix with no configuration

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{3.3}$$

Then $n \le \frac{3}{2}m+1$ and this is best possible.

Proof. Note that if α, β are two columns of A with the column sum of α less than the column sum of β then the forbidden configurations force $\alpha \le \beta$. Also if A_i is the submatrix of A (on m rows) of all columns of column sum i then A_i has no configuration F_1 in (2.8) and so, as in Lemma 2.7, the columns form a type 1 or type 2 pattern, hence there are at most m columns in A_i. In consideration of the bound we may assume there are columns of two column sums i, j, $1 \le i < j$. Using the covering relation of columns of column sum j over those of column sum i, we can decompose A, by permuting rows and columns as follows:

$$A = \begin{bmatrix} B & J \\ O & C \end{bmatrix}, \tag{3.4}$$

where B is an $s\times n'$ (0,1)-matrix of all columns of A of column sum at most i , 0 denote an $(m-s)\times n'$ matrix of 0's, C is $(m-s)\times n''$ matrix which with the $s\times n''$ matrix J of all 1's consists of those columns of A of column sum greater than i. By induction applied to B and C we get $n' \le \frac{3}{2} s + 1$ and $n'' \le \frac{3}{2}(m-s) + 1$. But if B has a column of 1's then C does not have a column of 0's and so either n' or n'' is one short of its bound. Hence

$$\begin{aligned} n = n' + n'' &\le (\tfrac{3}{2}s + 1) + (\tfrac{3}{2}(m-s)+1) - 1 \\ &\le \tfrac{3}{2}m + 1 \end{aligned} \tag{3.5}$$

The bound is seen to best possible by taking $m = 2k$ and then k copies of

$$\begin{bmatrix} 0\,0\,1 \\ 0\,1\,0 \end{bmatrix} \tag{3.6}$$

in a diagonal fashion to form a $m\times\frac{3}{2}m$ matrix by putting 0's below those blocks and 1's above and then adding the column of 1's. □

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