# Forbidden Configurations: A Survey

Richard Anstee
UBC, Vancouver

University of Ottawa
Carleton University
Jan. 15, 2010

# Introduction

Forbidden configurations are first described as a problem area in a 1985 paper. My subsequent work has involved a number of coauthors: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well (e.g. Balachandran, Dukes). For example, the definition of *VC-dimension* uses a forbidden configuration. The notion of *trace* is the set theory name for a configuration.

Survey at www.math.ubc.ca/∼anstee

**Definition** We say that a matrix $A$ is *simple* if it is a (0,1)-matrix with no repeated columns.

**Definition** We say that a matrix $A$ is *simple* if it is a (0,1)-matrix with no repeated columns.
i.e. if $A$ is $m$-rowed then $A$ is the incidence matrix of some $\mathcal{A} \subseteq 2^{[m]}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\}$$

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

We consider the property of forbidding a configuration $F$ in $A$ for which we say $F$ is a *forbidden configuration* in $A$.

**Definition** Let $\text{forb}(m, F)$ be the largest function of $m$ and $F$ so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration $F$. Thus if $A$ is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then $A$ contains $F$ as a configuration.

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & & 0 & \boxed{0} \end{bmatrix} = A$$

We consider the property of forbidding a configuration $F$ in $A$ for which we say $F$ is a *forbidden configuration* in $A$.

**Definition** Let $\text{forb}(m, F)$ be the largest function of $m$ and $F$ so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration $F$. Thus if $A$ is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then $A$ contains $F$ as a configuration.

For example, $\text{forb}(m, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 2$.

For example, $\quad \mathrm{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

For example, $\text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m-1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For example, $\text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m-1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Each pair of rows of $A$ can have at most 2 configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ else if it had 3 it would have 2 in the same orientation and the forbidden configuration.

For example, $\quad \text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m - 1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Each pair of rows of $A$ can have at most 2 configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ else if it had 3 it would have 2 in the same orientation and the forbidden configuration.

Thus if $n$ denotes the number of columns not all 0's or all 1's, then

$$(m-1)n \leq 2\binom{m}{2}$$

from which we deduce $n \leq m$ and hence the bound.

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

**Corollary** *Let $F$ be a $k \times \ell$ simple matrix. Then* $forb(m, F) = O(m^{k-1})$

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

**Corollary** *Let $F$ be a $k \times \ell$ simple matrix. Then* $forb(m, F) = O(m^{k-1})$

**Theorem** (Füredi 83). *Let $F$ be a $k \times \ell$ matrix. Then* $forb(m, F) = O(m^k)$

**Definition** Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with $k$ 1's on top of $\ell$ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

**Definition** Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with $k$ 1's on top of $\ell$ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

**Definition** Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of $t$ copies of $M$ placed side by side.

**Definition** Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with $k$ 1's on top of $\ell$ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

**Definition** Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of $t$ copies of $M$ placed side by side.

**Theorem** (A, Füredi 86) *Let $t, k$ be given.*

$$forb(m, t \cdot K_k) \leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

*with equality if a certain $k$-design exists.*

**Definition** Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with $k$ 1's on top of $\ell$ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

**Definition** Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of $t$ copies of $M$ placed side by side.

**Theorem** (A, Füredi 86) *Let $t, k$ be given.*

$$forb(m, t \cdot K_k) \leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

*with equality if a certain $k$-design exists.*

**Definition** Let $K_k^\ell$ denote the $k \times \binom{k}{\ell}$ simple matrix of all possible columns of sum $\ell$ on $k$ rows.

# Critical Substructures

**Definition** A *critical substructure* of a configuration $F$ is a minimal configuration $F'$ contained in $F$ such that

$$\text{forb}(m, F) = \text{forb}(m, F')$$

A critical substructure is what drives the construction yielding a lower bound $\text{forb}(m, F)$ where some other argument provides the upper bound for $\text{forb}(m, F)$.
A consequence is that for a configuration $F''$ which contains $F'$ and is contained in $F$, we deduce that

$$\text{forb}(m, F) = \text{forb}(m, F'') = \text{forb}(m, F')$$

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{|c|ccccccc} \color{red}{1} & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \color{red}{1} & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \color{red}{1} & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{c|ccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{cccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{ccccccc|c|} 1 & 1 & 1 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\color{red}{\mathbf{0}_3}$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{cc|cccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Critical Substructures for $K_3$

$$K_3 = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, $K_3^2$, $K_3^1$, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

# Designs and Forbidden Configurations

A 2-design $S_\lambda(2,3,v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v]=\{1,2,\ldots,v\}$ such that for each pair $i,j\in\binom{[v]}{2}$, there are exactly $\lambda$ triples containing $i,j$. If we encode the triple system as a $v$-rowed (0,1)-matrix $A$ such that the columns are the incidence vectors of the triples, then $A$ has no $2\times(\lambda+1)$ submatrix of 1's.

# Designs and Forbidden Configurations

A 2-design $S_\lambda(2,3,v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v]=\{1,2,\ldots,v\}$ such that for each pair $i,j\in\binom{[v]}{2}$, there are exactly $\lambda$ triples containing $i,j$. If we encode the triple system as a $v$-rowed (0,1)-matrix $A$ such that the columns are the incidence vectors of the triples, then $A$ has no $2\times(\lambda+1)$ submatrix of 1's.

**Remark** If $A$ is a $v\times n$ (0,1)-matrix with column sums 3 and $A$ has no $2\times(\lambda+1)$ submatrix of 1's then $n\le\frac{\lambda}{3}\binom{v}{2}$ with equality if and only if the columns of $A$ correspond to the triples of a 2-design $S_\lambda(2,3,v)$.

**Theorem** (A, Barekat) Let $\lambda$ and $v$ be given integers. There exists an $M$ so that for $v > M$, if $A$ is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-1\}$ and $A$ has no $3 \times (\lambda+1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

and we have equality if and only if the columns of $A$ correspond to the triples of a 2-design $S_\lambda(2, 3, v)$.

**Theorem** (A, Barekat) Let $\lambda$ and $v$ be given integers. There exists an $M$ so that for $v > M$, if $A$ is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and $A$ has no $4 \times (\lambda+1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers $a, b$ with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of $A$ of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of $A$ of column sum $v - 3$ corresponding to $(v-3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

**Theorem** (N. Balachandran 09) Let $\lambda$ and $v$ be given integers. There exists an $M$ so that for $v > M$, if $A$ is an $v \times n$ (0,1)-matrix with column sums in $\{4, 5, \ldots, v-1\}$ and $A$ has no $4 \times 2$ configuration

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$n \leq \frac{1}{4}\binom{v}{3}$$

with equality only if there is 3-design $S_1(3, 4, v)$. The $\frac{1}{4}\binom{v}{3}$ columns of $A$ have column sum 4 and correspond to 4-sets of the 3-design $S_1(3, 4, v)$.

Naranjan Balachandran has indicated that he has made further progress on this problem

# Exact Bounds

A, Barekat 09

| Configuration F | Exact Bound forb$(m, F)$ |
| --- | --- |
| $\overbrace{\begin{bmatrix}1\,1\cdots1\\1\,1\cdots1\\0\,0\cdots0\end{bmatrix}}^{p}$ | $\frac{p+1}{3}\binom{m}{2}+\binom{m}{1}+2\binom{m}{0}$ for $m$ large, $m\equiv 1,3(\text{mod}6)$ |
| $\overbrace{\begin{bmatrix}1\,1\cdots1\\1\,1\cdots1\\0\,0\cdots0\\0\,0\cdots0\end{bmatrix}}^{p}$ | $\frac{p+3}{3}\binom{m}{2}+2\binom{m}{1}+2\binom{m}{0}$ for $m$ large, $m\equiv 1,3(\text{mod}6)$ |

# Another Example of Critical Substructures

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Theorem** (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

# Another Example of Critical Substructures

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Theorem** (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

# Another Example of Critical Substructures

$$F_1 = \left[\begin{array}{cc|cc|} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

**Theorem** (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \left[\begin{array}{cc|cc|} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

# Exact Bounds

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

**Theorem** (A,Karp 09)

$$\text{forb}(m, F) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \leq \frac{4}{3}\binom{m}{2} + m + 1$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

# Exact Bounds

$$F_3 = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

**Theorem** (A,Karp 09)

$$\text{forb}(m, F) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \leq \frac{4}{3}\binom{m}{2} + m + 1$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

# Exact Bounds

A, Griggs, Sali 97, A, Ferguson, Sali 01, A, Kamoosi 07
A, Barekat, Sali 09, A, Barekat 09, A, Karp 09

| **Configuration F** | **Exact Bound forb**$(m, F)$ |
|---|---|
| $\begin{bmatrix}0\\1\end{bmatrix}$ | $2$ |
| $\begin{bmatrix}0&0\\1&1\end{bmatrix}$ | $m+2$ |
| $\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}$ | $2m+2$ |
| $\begin{bmatrix}0&0&0&0\\1&1&1&1\end{bmatrix}$ | $\left\lfloor \frac{5m}{2} \right\rfloor + 2$ |
| $q \cdot \begin{bmatrix}0\\1\end{bmatrix}$ | $\left\lfloor \frac{(q+1)m}{2} \right\rfloor + 2, \quad$ for $m$ large |

# Exact Bounds

| Configuration F | Exact Bound forb$(m, F)$ |
|---|---|
| $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{3m}{2} \right\rfloor + 1$ |
| $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{7m}{3} \right\rfloor + 1$ |
| $\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{11m}{4} \right\rfloor + 1$ |
| $\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{15m}{4} \right\rfloor + 1$ |

# Exact Bounds

| Configuration F | Exact Bound forb$(m, F)$ |
|---|---|
| $\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{8m}{3} \right\rfloor$ |
| $\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{bmatrix}$ | $\left\lfloor \frac{10m}{3} - \frac{4}{3} \right\rfloor$ |
| $\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$ | $4m$ |
| $\begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{p}\\ 0\cdots0\,1\cdots1\end{bmatrix}$ | $pm - p + 2$ |

# $k \times 2$ Forbidden Configurations

$$\text{Let } F_{abcd} = \begin{array}{c} a\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ b\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ c\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ d\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \end{array} \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

For the purposes of forbidden configurations we may assume that $a \geq d$ and $b \geq c$.

The following result used a difficult 'stability' result and the resulting constants in the bounds were unrealistic but the asymptotics agree with a general conjecture.

**Theorem** (A-Keevash 06) *Assume a,b,c,d are given with* $a \geq d$ *and* $b \geq c$*. If* $b > c$ *or* $a, b \geq 1$*, then*

$$forb(m, F_{abcd}) = \Theta(m^{a+b-1}).$$

*Also* $forb(m, F_{0bb0}) = \Theta(m^b)$ *and* $forb(m, F_{a00d}) = \Theta(m^a)$*.*

Note that the first column of $F_{abcd}$ is $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$.

**Theorem** (A, Karp 09) Let $a, b \geq 2$. Then

$$\text{forb}(m, F_{ab01}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab10}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab11}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_2) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m-1}^{m} \binom{m}{j}$$

**Problem** (A, Karp 09). Let $a, b, c, d$ be given with $a, b$ much larger than $c, d$. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

**Problem** (A, Karp 09). Let $a, b, c, d$ be given with $a, b$ much larger than $c, d$. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

We are asking when $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$ is a critical substructure for $F_{abcd}$.

# Pseudo-Exact Bounds

When determining forb$(m, F)$ it is possible that there is a subconfiguration that dominates the bound but does not yield the exact bound? This is often the case (when the bound is known) but the following result sharpens known results.

**Theorem** (A, Raggi 09) Let $t, q \geq 1$ be given. Let

$$F_4(t,q) = \left[ t \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} q \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right].$$

Then forb$(m, F_4(t,q))$ is forb$(m, t \cdot \mathbf{1}_4)$ plus $O(qm^2)$.

$$F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

**Theorem** *Let $c$ be a positive real number. Let $A$ be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no $F_{2110}$. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no $F_{2110}$.*

**Theorem** (P. Dukes 09) $\text{forb}(m, F_{2110}) \leq .691m^2$

The proof used inequalities and linear programming

$$F_{0220} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

**Theorem** (A, Barekat, Sali 09)

$$\text{forb}(m, F_{0220}) = \binom{m}{2} + m - 2$$

$$F_{0220} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

**Theorem** (A, Barekat, Sali 09)

$$\text{forb}(m, F_{0220}) = \binom{m}{2} + m - 2$$

**Conjecture** *forb*$(m, t \cdot F_{0220})$ *is* $O(m^2)$.

The result is true for $t = 2$. The result would follow from the general conjecture

# Two interesting examples

Let $$F_5 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_6 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{forb}(m, F_5) = 2m, \qquad \text{forb}(m, F_6) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

# Two interesting examples

Let $$F_5 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_6 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{forb}(m, F_5) = 2m, \qquad \text{forb}(m, F_6) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

**Problem** *What drives the asymptotics of forb$(m, F)$? What structures in $F$ are important?*

# Refinements of the Sauer Bound

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*$(m, K_k)$ *is* $\Theta(m^{k-1})$.

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

**Theorem** (A, Fleming) *Let $F$ be a $k \times \ell$ simple matrix such that there is a pair of rows with no configuration $E_1$ and there is a pair of rows with no configuration $E_2$ and there is a pair of rows with no configuration $E_3$. Then forb$(m, F)$ is $O(m^{k-2})$.*

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*$(m, K_k)$ *is* $\Theta(m^{k-1})$.

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

**Theorem** (A, Fleming) *Let* $F$ *be a* $k \times \ell$ *simple matrix such that there is a pair of rows with no configuration* $E_1$ *and there is a pair of rows with no configuration* $E_2$ *and there is a pair of rows with no configuration* $E_3$. *Then forb*$(m, F)$ *is* $O(m^{k-2})$.

Note that $F_7 = \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 0\\0 & 1 & 0 & 1 & 0 & 1\\1 & 1 & 0 & 0 & 1 & 1\\1 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$ has no $E_1$ and no $E_2$ on rows 1,2 and no $E_3$ on rows 3,4. Thus forb$(m, F_7)$ is $O(m^2)$.

# New Result

$$F_7(t) = \left[ \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

**Theorem** (A, Raggi, Sali 09) Let $t$ be given. Then $\mathrm{forb}(m, F_7(t))$ is $O(m^2)$.

Note that $F_7 = F_7(1)$. We cannot maintain the quadratic bound and repeat any other columns of $F_7$ since repeating columns of sum 1 or 3 in $F_7$ will yield constructions of $\Theta(m^3)$ columns avoiding them.

**Definition** $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

**Theorem** (A, Fleming) *Let $E$ be given with $E \in \{E_1, E_2, E_3\}$. Let $F$ be a $k \times \ell$ simple matrix with the property that every pair of rows contains the configuration $E$. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} \boxed{\begin{matrix}1 & 0\\0 & 1\end{matrix}} & \begin{matrix}1 & 0\\0 & 1\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \begin{matrix}1 & 1\end{matrix}\end{bmatrix}$$ has $E_3$ on rows 1,2.

**Definition** $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**Theorem** (A, Fleming) *Let $E$ be given with $E \in \{E_1, E_2, E_3\}$. Let $F$ be a $k \times \ell$ simple matrix with the property that every pair of rows contains the configuration $E$. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \boxed{1 \quad 0} & & 1 \\ 0 & \boxed{0 \quad 1} & & 1 \end{bmatrix}$$ has $E_3$ on rows 2,3.

**Definition** $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

**Theorem** (A, Fleming) *Let $E$ be given with $E \in \{E_1, E_2, E_3\}$. Let $F$ be a $k \times \ell$ simple matrix with the property that every pair of rows contains the configuration $E$. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix}\boxed{1} & 0 & 1 & \boxed{0}\\ 0 & 1 & 0 & 1\\ \boxed{0} & 0 & 1 & \boxed{1}\end{bmatrix}$$ has $E_3$ on rows 1,3.

**Definition** $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

**Theorem** (A, Fleming) *Let $E$ be given with $E \in \{E_1, E_2, E_3\}$. Let $F$ be a $k \times \ell$ simple matrix with the property that every pair of rows contains the configuration $E$. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} \boxed{1} & 0 & 1 & \boxed{0} \\ 0 & 1 & 0 & 1 \\ \boxed{0} & 0 & 1 & \boxed{1} \end{bmatrix}$$ has $E_3$ on rows 1,3.

Note that $F_6$ has $E_3$ on every pair of rows hence forb$(m, F_6)$ is $\Theta(m^2)$ (A, Griggs, Sali 97).

# A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

# A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

Note that $\text{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

**Definition** Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

**Definition** Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

# The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of $p$ such that $F$ is a configuration in every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

# The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of $p$ such that $F$ is a configuration in every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

**Conjecture** (A, Sali 05) *forb*$(m, F)$ *is* $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

# The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of $p$ such that $F$ is a configuration in every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

**Conjecture** (A, Sali 05) *forb*$(m, F)$ *is* $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

The conjecture has been verified for $k \times \ell$ $F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

Let $B$ be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices $C, D$, let $C \backslash D$ denote the matrix obtained from $C$ by deleting any columns of $D$ that are in $C$ (i.e. set difference). Let $F_B(t) = [K_k | t \cdot [K_k \backslash B]]$. For $k = 4$ an example is

$$\left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]\right]$$

Let $B$ be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices $C, D$, let $C \backslash D$ denote the matrix obtained from $C$ by deleting any columns of $D$ that are in $C$ (i.e. set difference). Let $F_B(t) = [K_k | t \cdot [K_k \backslash B]]$. For $k = 4$ an example is

$$\left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]\right]$$

**Theorem** (A, Griggs, Sali 97, A, Sali 05,
A, Fleming, Füredi, Sali 05)
*forb*$(m, F_B(t))$ *is* $\Theta(m^{k-1})$.

The difficult problem here was the bound although induction works.

Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \; (t+1) \cdot \left[ \begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k\ (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \ (t+1) \cdot \left[ \begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

**Theorem** ( A, Sali 05 (for $k = 3$), A, Fleming 09)
*forb*$(m, F_D(t))$ *is* $\Theta(m^{k-1})$.

The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \; (t+1) \cdot \left[ \begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

**Theorem** ( A, Sali 05 (for $k = 3$), A, Fleming 09)
*forb*$(m, F_D(t))$ *is* $\Theta(m^{k-1})$.

The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

**Theorem** *Let $k$ be given and assume $F$ is a $k$-rowed configuration which is not a configuration in $F_B(t)$ for any choice of $B$ as a $k \times (k+1)$ simple matrix with one column of each column sum and not in $F_D(t)$ or $F_D(t)^c$, for any $t$. Then forb$(m, F)$ is $\Theta(m^k)$.*

THANKS FOR THE CHANCE TO TALK IN OTTAWA!

# Exact Bounds

$$F_5 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Theorem** (A, Dunwoody) $\text{forb}(m, F_5) = \lfloor \frac{m^2}{4} \rfloor + m + 1$

**Proof:** The proof technique is that of shifting, popularized by Frankl. A paper of Alon 83 using shifting refers to the possibility of such a result.

**Definition** We say $\mathcal{F} \subseteq 2^{[m]}$ is *t-intersecting* if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$.

**Theorem** (Ahlswede and Khachatrian 97)
*Complete Intersection Theorem.*
*Let $k, r$ be given. A maximum sized $(k\text{-}r)$-intersecting $k$-uniform family $\mathcal{F} \subseteq \binom{[m]}{k}$ is isomorphic to $\mathcal{I}_{r_1,r_2}$ for some choice $r_1 + r_2 = r$ and for some choice $G \subseteq [m]$ where $|G| = k - r_1 + r_2$ where*
$\mathcal{I}_{r_1,r_2} = \{A \subseteq \binom{[m]}{k} : |A \cap G| \geq k - r_1\}$

This generalizes the Erdős-Ko-Rado Theorem (61).

**Theorem** (A-Keevash 06) Stability Lemma.
*Let* $\mathcal{F} \subseteq \binom{[m]}{k}$. *Assume that* $\mathcal{F}$ *is* $(k\text{-}r)$*-intersecting and*

$$|\mathcal{F}| \geq (6r)^{5r+7} m^{r-1}.$$

*Then* $\mathcal{F} \subseteq \mathcal{I}_{r_1,r_2}$ *for some choice* $r_1 + r_2 = r$ *and for some choice* $G \subseteq [m]$ *where* $|G| = k - r_1 + r_2$.

This result is for large intersections; we use it with a fixed $r$ where $k$ can grow with $m$.