Math 523 Assignment #3 Due Friday Feb 25

1.

a) Explain why you do not expect a strongly polynomial algorithm for PRIME.

b) Explain how a pivot rule for the simplex method that only requires a number of pivots polynomial in the number of constraints (m) and the number of variables (n) could yield a strongly polynomial algorithm for LP.

2. Given a digraph $D = (N, E)$ we can form a matrix $A = (a_{ij})$ of $|N|$ rows and $|E|$ columns as follows.

$$a_{ij} = \begin{cases} 1 & \text{arc } j \text{ has tail } i \\ -1 & \text{arc } j \text{ has head } i \\ 0 & \text{otherwise} \end{cases}$$

Show than any square submatrix of A has determinant -1 or 0 or 1. (Hence in our LP application, B^{-1} is an integral matrix).

3. (Car Leasing) Assume you are running a business and wish to lease cars for the next six months with a need for d_i cars in month i. There are three choices for the lease: a one month lease at a cost of \$a, a two month lease at a cost of \$b and a three month lease at a cost of \$c. You may assume $a > b/2 > c/3$. We can formulate this as a minimum cost circulation problem by having 7 nodes $1, 2, 3, 4, 5, 6, 7$ with arcs $(i, i+1)$ for $i = 1, 2, 3, 4, 5, 6$ with lower bound d_i (and upper bounds very large and costs 0) and also arcs $(i, i-1)$ for $i = 2, 3, 4, 5, 6, 7$ with cost a (and lower bounds 0 and upper bounds large) and arcs $(i, i-2)$ for $i = 3, 4, 5, 6, 7$ with cost b (and lower bounds 0 and upper bounds large) and arcs $(i, i-3)$ for $i = 4, 5, 6, 7$ with cost c (and lower bounds 0 and upper bounds large). Convince me that we can use this to find a minimum cost way of leasing cars subject to the needs.

4. Let $G = (V, E)$ be a connected graph with no multiple edges or loops and let $x, y \in V$ be given. We are interested in finding the shortest path from x to y that uses an even number of edges. Consider the following graph H. Roughly speaking H consists of two copies of G where each pair of vertices the same in G are joined in H and then we have x deleted from the first copy and y deleted from the second copy. Let $V' = \{v' : v \in V\}$. Then $V(H) = (V \backslash x) \cup (V' \backslash y')$. We form the edges of H to consist of three types of edges: $\{(a, b) : a, b \in V \backslash x, (a, b) \in E\}$, $\{(a', b') : a', b' \in V' \backslash y', (a, b) \in E\}$ and the cross edges $\{(a, a') : a \in V, a' \in V'\}$. By considering the possible structure of perfect matchings in terms of cross edges and edges of G either in the original copy or in the copy with 's, show that H has perfect matching if and only if G has an x-y-path in G that uses an even number of edges. Now by imposing edge weights of 1 on the edges

$$\{(a, b) : a, b \in V \backslash x, (a, b) \in E\} \cup \{(a', b') : a', b' \in V' \backslash y', (a, b) \in E\}$$

and 0 on the cross edges show that a minimum weight perfect matching in H solves the problem of determining the shortest x-y-path in G that uses an even number of edges. What should you do for the shortest x-y-path that uses an odd number of edges? Also explain how to use the idea to find the odd cycle in G of least weight if you were given edge weights $w(e) \geq 0$ for $e \in E$.