MATH 443 Various Notations

In this course, a *graph* G consists of a finite set $V(G) = V$ of *vertices* and a finite set $E(G) = E$ of *edges* such that each edge $e = uv$ has associated with it two endpoints $u, v \in V$ which need not be distinct. A *loop* is an edge with both endpoints the same e.g. $e = vv$.

A *simple* graph G (or just a graph G) has no multiple edges or loops. We refer to a *multigraph* if we allow multiple edges and a general graph is we also allow loops.

A *walk* of *length* k is a sequence $v_0e_1v_1e_2v_2 \cdots e_kv_k$ such that $v_i \in V(G)$, $e_i \in E(G)$ and the endpoints of e_i are v_{i-1},v_i or in other words $e_i = v_{i-1}v_i$. We say that W is a v_0-v_k-walk.

A *trail* is a walk with no repeated edges.

A *path* is a walk with no repeated vertices.

A u-v-walk (respectively trail or path) is a walk (respectively trail or path) with first vertex u and last vertex v.

A walk (respectively trail) is *closed* if it has at least one edge and the first and last vertices of the walk (resp. trail) are the same. In our example $v_0 = v_k$.

A graph is *connected* if each pair of vertices are joined by a walk. A *component* of a graph is a maximal (with respect to vertices) connected subgraph of G. We can think of a component as an equivalence class of vertices where we have an equivalence relation that says that x is related to y if there is an x-y-walk in G. For a connected graph G, a *cut edge* is an edge e for which $G\backslash e$ is disconnected.

A directed graph is *strongly connected* if for each pair of vertices x, y there is both a directed x-y-path as well as a directed y-x-path.

An *eulerian circuit* is a closed trail in which each edge of G is used. We typically allow a general graph in this problem.

A *cycle* is a closed trail in which the only pair of repeated vertices is the first and last vertices of the trail. Our definition of C_n refers to the isomorphism class of cycles of n edges. Note that a loop is a cycle and is of course a closed walk or trail. Also if we have two vertices x, y joined by an edge e then $xeyex$ is a closed walk but not a cycle. If we have two edges e, f with endpoints x, y then we get a cycle $xeyfx$. A *chord* of a cycle is an edge joining two vertices of the cycle not already joined by edges of the cycle.

A graph is *bipartite* if the vertices $V(G)$ can be partitioned into X, Y ($X \cup Y = V(G)$, $X \cap Y = \emptyset$) so that for each edge $e \in E(G)$, one endpoint is in X and one endpoint is in Y.

A *subgraph* H of G is a graph for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Of course in order for H to be a graph, for each edge $e \in E(H)$ we must have both endpoints in $V(H)$.

An *induced subgraph* $H = (V(H), E(H))$ is a subgraph for which each edge e of G where both endpoints are in $V(H)$ is also in $E(H)$. We use the notation $H = G[V(H)]$. We refer to a subgraph as induced even if we haven't specified $V(H)$ and in that case are asserting the existence of an appropriate $V(H)$.

A subgraph H of G is called a *spanning subgraph* if $V(H) = V(G)$. A spanning cycle is called a *hamiltonian cycle*.

A *tree* is a subgraph which is connected and has no cycles. We had alternate definitions in class. Note that a tree on more than one vertex must have a vertex of degree one.

A *spanning tree* is a spanning subgraph which is a tree and is also spanning.

A *matching* is a set $M \subseteq E(G)$ of edges no two of which are incident. A *perfect matching* is a matching so for each vertex $v \in V(G)$ there is an edge of M which is incident to v. A spanning 1-regular subgraph is called a 1-factor and the edges form a perfect matching and this is then the same as a perfect matching. If we have a vector $f = (f(v) : v \in V)$, then an f-factor is a spanning subgraph of G with degree at vertex v equal to $f(v)$ for each $v \in V$.

A *clique* is a set of vertices for which each pair are joined by edges; i.e. a set S is a clique in G if the subgraph induced by S is $K_{|S|}$.

An *independent set* of vertices is a set S of vertices for which no pair are joined by edges. Thus an independent set corresponds to a clique in G^c.

The *Line Graph* $L(G)$ is a graph obtained from G with $L(G) = (E(G), E')$ where for $e_1, e_2 \in E(G)$, $e_1e_2 \in E'$ if the have a vertex in common.

Graph parameters

1. The *degree* $d_G(v)$ (or just $d(v)$) of a vertex v is the number of endpoints of edges equal to v, hence the number of incidences of edges with v noting that we count a loop for two incidences. The degree sequence $d_1, d_2, \ldots$ of a graph (with $d_i = d_G(i)$), typically has $d_1 \geq d_2 \geq \cdots$.

2. A graph is *cubic* if every degree is 3. A graph is *r-regular* if every degree is r. If a vector $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ is given, then an f-factor is a subgraph $\mathbf{x} = (x(e) : e \in E(G))$ of G satisfying $x(e) \in \{0, 1\}$ with $d_{\mathbf{x}}(i) = \sum_{e \text{ hits } i} x(e)$ being the associated degree. A fractional f-factor is a vector $\mathbf{x} = (x(e) : e \in E(G))$ of G satisfying $x(e) \in [0, 1]$ with $d_{\mathbf{x}}(i) = \sum_{e \text{ hits } i} x(e)$.

3. We define $\delta(G) = \min_{v \in V} d(v), \qquad \Delta(G) = \max_{v \in V} d(v)$

4. We have distance function $d_G(x, y) = d(x, y)$ being the length of shortest x-y-path

5. The *diameter* $\operatorname{diam}(G) = \max_{x,y \in V} d(x, y)$.

6. $\kappa(G)$ is the (vertex) connectivity of G and is the minimum number of vertices that must be deleted from G to either disconnect the graph or leave a single vertex. A graph is k-connected if $\kappa(G) \geq k$. A *cut* in a graph is a set of vertices S such that $G - S$ is disconnected.

7. $\kappa'(G)$ is the edge connectivity of G and is the minimum number of edges that must be deleted from G to disconnect the graph. A graph is k-edge-connected if $\kappa'(G) \geq k$. An *edge cut* in a graph is a set of edges of the form $[S, V \backslash S]$ where S is a set of vertices $S \neq \emptyset, V(G)$. Then $G \backslash [S, V(G) \backslash S]$ is disconnected.

8. The *girth* of a graph is the length of the shortest cycle in a graph.

9. $\omega(G)$ is the cardinality of the largest clique.

10. $\alpha(G)$ is the cardinality of the largest independent set ($\alpha(G) = \omega(G^c)$). Also known as stable sets.

11. $\chi(G)$ is the minimum number of colours in a vertex colouring of G. The *chromatic polynomial* of G $\chi(G; k)$ is and gives the number of colourings of G that are possible with k colours.

12. $\chi'(G)$ is the minimum number of colours in a edge colouring of G.

13. $\tau(G)$ is the number of (vertex labelled) spanning trees of G.

14. $R_r(k_1, k_2, \ldots k_\ell)$ is the smallest n so that every colouring of the r-sets of $\{1, 2, \ldots, n\}$ has the property that there is a colour i and a set of vertices X with $|X| = k_i$ such that all r-sets contained in X have colour i (a version of a monochromatic clique when generalizing from $r = 2$ to general r). These Ramsey numbers exist.

Special Graphs

1. K_n denotes the simple graph on n vertices with every pair of vertices joined by an edge. It is called the *complete* graph.

2. C_n denotes the simple graph on n vertices $\{1, 2, \ldots, n\}$ with $E(C_n) = \{12, 23, \ldots (n-1)n, n1\}$. It is called the *cycle* of length n.

3. P_n denotes the simple graph on n vertices $\{1, 2, \ldots, n\}$ with $E(P_n) = \{12, 23, \ldots (n-1)n\}$. It is called the *path* of length $n-1$ since it has $n-1$ edges.

4. $K_{r,s}$ denotes the simple graph on $r+s$ vertices where $|X| = r$ and $|Y| = s$ and for each choice $x \in X$ and $y \in Y$ we have $xy \in E(K_{r,s})$. It is called the *complete bipartite* graph on parts of size r and s.

5. Q_k is the k-dimensional hypercube consisting of 2^k vertices, each vertex corresponding to a different (0,1)-string of length k and we join two vertices if their associated strings differ in exactly one position.

6. The *Petersen graph* (which has no special symbol) is the graph on 10 vertices with the property that each vertex has degree 3 and each pair of vertices are either joined by an edge or joined by a unique path of two edges but not both (i.e. no C_3 or C_4 subgraph).

7. W_n is *wheel graph* on n vertices with a central vertex joined to all others and $n-1$ vertices whose induced subgraph is C_{n-1}.