MATH 441: Branch and Bound

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First we consideri a mixed integer program (MIP) which is an LP with certain variables restricted to be integers. In this problem any variables restricted to be integers are in fact forced to be (0,1)-variables.

If we have an Mixed Integer LP, then any version of the problem under consideration, but with fewer variables restricted to be integer than specified in the original, is considered to be a (partially) **relaxed problem**. In our version, any variable x_i that we have **relaxed** is still required to satisfy the equation $0 \le x_1 \le 1$. Consider the following LP.

$$\begin{array}{rrrrrrrr}
\text{maximize} & 10x_1 & +7x_2 & +4x_3 & +3x_4 & +x_5 & & \\
 & 2x_1 & +6x_2 & +x_3 & & & +x_6 & \le 7 \\
 & 2x_1 & -3x_2 & +4x_3 & +x_4 & +x_5 & & \le 3 \\
 & x_1 & & +2x_3 & -3x_4 & +x_5 & -x_6 & \le -1 \\
 & x_1 & +x_2 & +x_3 & +x_4 & -x_5 & & \le 3
\end{array}$$
$$x_j \in \{0,1\} \text{ for } j = 1,2,3,4,5, \quad x_6 \ge 0$$

(1) (Completely) Relaxed LP

$$\begin{array}{rrrrrrrr}
\text{maximize} & 10x_1 & +7x_2 & +4x_3 & +3x_4 & +x_5 & & \\
 & 2x_1 & +6x_2 & +x_3 & & & +x_6 & \le 7 \\
 & 2x_1 & -3x_2 & +4x_3 & +x_4 & +x_5 & & \le 3 \\
 & x_1 & & +2x_3 & -3x_4 & +x_5 & -x_6 & \le -1 \\
 & x_1 & +x_2 & +x_3 & +x_4 & -x_5 & & \le 3
\end{array}$$
$$0 \le x_j \le 1 \text{ for } j = 1,2,3,4,5, \quad x_6 \ge 0$$

The optimal solution is $z^* = 20.21$, $x^* = (1, 0.72, 0.49, 1, 0.21, 0.19)$.

Branch on x_5 (closest to 0 or 1)

(2) $x_5 = 0$ **(or** $x_5 \le 0$**)** $\quad 0 \le x_j \le 1$ for $j = 1,2,3,4, \quad x_6 \ge 0$

The optimal solution is $z^* = 19.45$, $x^* = (1, 0.78, 0.33, 0.89, 0, 0)$.

(3) $x_5 = 1$ **(or** $x_5 \ge 1$**)** $\quad 0 \le x_j \le 1$ for $j = 1,2,3,4, \quad x_6 \ge 0$

The optimal solution is $z^* = 19.97$, $x^* = (1, 0.7, 0.27, 1, 1, 0.55)$.

The current candidate list is (2), (3).

Select (3) and branch on x_3

(4) $x_5 = 1$**,** $x_3 = 0 \quad 0 \le x_j \le 1$ for $j = 1,2,4, \quad x_6 \ge 0$

The optimal solution is $z^* = 19.83$, $x^* = (1, 0.83, 0, 1, 1, 0)$.

(5) $x_5 = 1$**,** $x_3 = 1 \quad 0 \le x_j \le 1$ for $j = 1,2,4, \quad x_6 \ge 0$

This LP is infeasible.

The current candidate list is (2), (4).

Select (4) and branch on x_2

(6) $x_5 = 0$**,** $x_3 = 0$**,** $x_2 = 0 \quad 0 \le x_j \le 1$ for $j = 1,4, \quad x_6 \ge 0$

The optimal solution is $z^* = 11$, $x^* = (1, 0, 0, 0, 1, 3)$.

(7) $x_5 = 0$**,** $x_3 = 0$**,** $x_2 = 1 \quad 0 \le x_j \le 1$ for $j = 1,4, \quad x_6 \ge 0$

The optimal solution is $z^* = 16$, $x^* = (0.5, 1, 0, 1, 1, 0)$.

The current candidate list is (2), (7) and the current best integer solution is (6) with $z^* = 11$.

Select (2) and branch on x_4

(8) $x_5 = 0$, $x_4 = 0$ $\quad 0 \le x_j \le 1$ for $j = 1, 2, 3$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 13.73$, $x^* = (1, 0.27, 0.45, 0, 0, 2.91)$.

(9) $x_5 = 0$, $x_4 = 1$ $\quad 0 \le x_j \le 1$ for $j = 1, 2, 3$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 19.4$, $x^* = (1, 0.8, 2, 1, 0, 0)$.

The current candidate list is (7), (8), (9) and the current best integer solution is (6) with $z^* = 11$.

Select (9) and branch on x_2

(10) $x_5 = 0$, $x_4 = 1$, $x_2 = 0$ $\quad 0 \le x_j \le 1$ for $j = 1, 3$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 13$, $x^* = (1, 0, 0, 1, 0, 0)$.

(11) $x_5 = 0$, $x_4 = 1$, $x_2 = 1$ $\quad 0 \le x_j \le 1$ for $j = 1, 3$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 15$, $x^* = (0.5, 1, 0, 1, 0, 0)$.

The current candidate list is (7), (8), (11) and the current best integer solution is (10) with $z^* = 13$.

Select (7) and branch on x_1

(12) $x_5 = 0$, $x_3 = 0$, $x_2 = 1$, $x_1 = 0$ $\quad 0 \le x_j \le 1$ for $j = 4$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 11$ so prune this branch.

(13) $x_5 = 0$, $x_3 = 0$, $x_2 = 1$, $x_1 = 1$ $\quad 0 \le x_j \le 1$ for $j = 4$, $\quad x_6 \ge 0$

This LP is infeasible.

The current candidate list is (8), (11) and the current best integer solution is (10) with $z^* = 13$.

Select (11) and branch on x_1

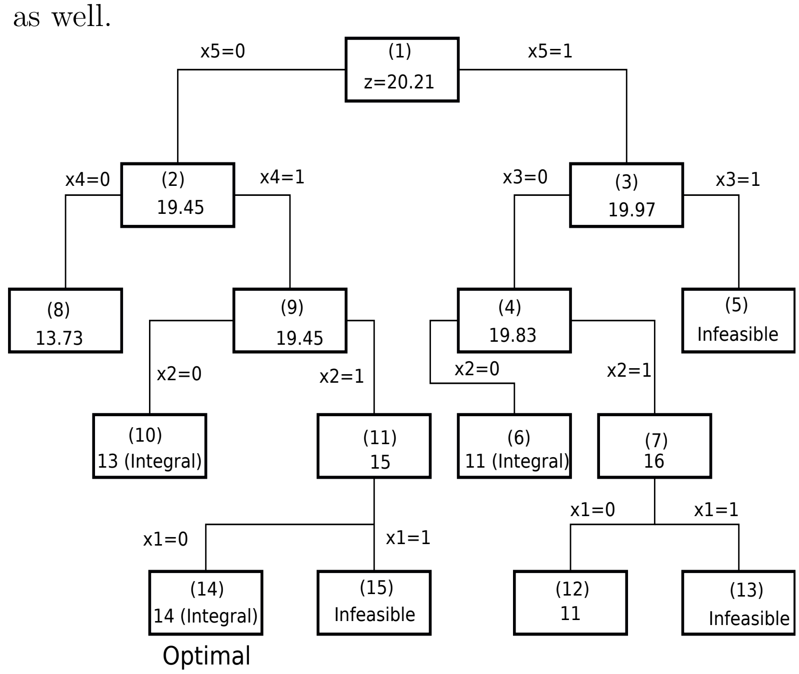
(14) $x_5 = 0$, $x_4 = 1$, $x_2 = 1$, $x_1 = 0$ $\quad 0 \le x_j \le 1$ for $j = 3$, $\quad x_6 \ge 0$

The optimal solution is $z^* = 14$, $x^* = (0, 1, 1, 1, 0, 0)$.

(15) $x_5 = 0$, $x_4 = 1$, $x_2 = 1$, $x_1 = 1$ $\quad 0 \le x_j \le 1$ for $j = 3$, $\quad x_6 \ge 0$

This LP is infeasible.

The current candidate list is (8) and the current best integer solution is (14) with $z^* = 14$ and so we can prune (8) as well.



I though I would try another example where the variables were not forced to be 0 or 1 but more general integer variables.

$$
\begin{array}{rrrrrl}
\text{maximize} & 21x_1 & +67x_2 & +77x_3 & +88x_4 & \\
 & x_1 & +5x_2 & +4x_3 & +7x_4 & \leq 30 \\
 & 2x_1 & +7x_2 & +2.2x_3 & +5x_4 & \leq 31 \\
 & 3x_1 & +6x_2 & +8x_3 & +8x_4 & \leq 55 \\
 & \multicolumn{4}{r}{x_j \geq 0 \text{ and } x_j \in \mathbf{Z} \text{ for } j = 1,2,3,4} &
\end{array}
$$

(1) (Completely) Relaxed LP

$$
\begin{array}{rrrrrl}
\text{maximize} & 21x_1 & +67x_2 & +77x_3 & +88x_4 & \\
 & x_1 & +5x_2 & +4x_3 & +7x_4 & \leq 30 \\
 & 2x_1 & +7x_2 & +2.2x_3 & +5x_4 & \leq 31 \\
 & 3x_1 & +6x_2 & +8x_3 & +8x_4 & \leq 55 \\
 & \multicolumn{4}{l}{x_j \geq 0 \text{ for } j = 1,2,3,4} &
\end{array}
$$

The optimal solution is $z^* = 540.9$, $x^* = (0, 1.25, 5.94, 0)$. THus we have node (1) with $z^* = 540.9$, $x^* = (0, 1.25, 5.94, 0)$.

Branch on x_3 (non-integer closest to an integer)

(2) add $x_3 \leq 5$

The optimal solution is $z^* = 531.66$, $x^* = (1.66, 1.66, 5, 0)$.

(3) add $x_3 \geq 6$ The optimal solution is $z^* = 540.16$, $x^* = (0, 1.6, 6, 0)$.

The current candidate list is (2), (3).

Select (3) and branch on x_2

(4) $x_2 \leq 1$**,** $x_3 \geq 6$

The optimal solution is $z^* = 540$, $x^* = (0, 1, 6, .13)$.

(5) $x_2 \geq 2$**,** $x_3 \geq 6$

This LP is infeasible.

The current candidate list is (2), (4).

Select (4) and branch on x_4

(6) $x_4 \leq 0$**,** $x_2 \leq 1$**,** $x_3 \geq 6$

The optimal solution is $z^* = 538.6$, $x^* = (0, 1, 6.13, 0)$.

(7) $x_4 \geq 1$**,** $x_3 = 0$**,** $x_2 = 1$

This LP is infeasible.

The current candidate list is (2), (6)

Select (6) and branch on x_3

(8) $x_3 \leq 6$**,** $x_4 \leq 0$**,** $x_2 \leq 1$**,** $x_3 \geq 6$

The optimal solution is $z^* = 536$, $x^* = (.33, 1, 6, 0)$.

(9) $x_3 \geq 7$**,** $x_4 \leq 0$**,** $x_2 \leq 1$**,** $x_3 \geq 6$

The current candidate list is (2), (8).

Select (8) and branch on x_1

(10) $x_1 \leq 0$**,** $x_3 \leq 6$**,** $x_4 \leq 0$**,** $x_2 \leq 1$**,** $x_3 \geq 6$

The optimal solution is $z^* = 529$, $x^* = (0, 1, 6, 0)$, which has all integer values.

(11) $x_1 \geq 1$**,** $x_3 \leq 6$**,** $x_4 \leq 0$**,** $x_2 \leq 1$**,** $x_3 \geq 6$

The optimal solution is $z^* = 527.6$, $x^* = (1, .66, 6, 0)$.

The current candidate list is (2), (10), (11) and the current best integer solution is (10) with $z^* = 529$. We *prune* (11) since any integer solution satisfyingthe inequalities of (11) will have $z \leq 527.6 < 529$.

Select (2) and branch on x_2

(12) $x_2 \leq 1$**,** $x_3 \leq 5$

The optimal solution is $z^* = 529.7$, $x^* = (1.77, 1, 5, .4)$.

(13) $x_2 \geq 2$**,** $x_3 \leq 5$

The optimal solution is $z^* = 524$, $x^* = (3, 2, 4.25, 0)$. We may prune this node.

The current candidate list is (12) and the current best integer solution is (10) with $z^* = 529$.

Select (12) and branch on x_1

(14) $x_1 \leq 1$**,** $x_2 \leq 1$**,** $x_3 \leq 5$

The optimal solution is $z^* = 523.3$, $x^* = (1, 1, 5, .57)$. We may prune this node.

(15) $x_1 \geq 2$**,** $x_2 \leq 1$**,** $x_3 \leq 5$

The optimal solution is $z^* = 528.37$, $x^* = (2, 1, 4.875, .5)$. We may prune this node.

The current candidate list is empty and so the current best integer solution is (10) with $z^* = 529$ and we know that it is optimal.