Proof of Weak Duality
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**Theorem** (Weak Duality) Let $\mathbf{x}^*$ be a feasible solution to the primal and let $\mathbf{y}^*$ be a feasible solution to the dual where

$$\begin{array}{llclll} & \max & \mathbf{c}\cdot\mathbf{x} & & \min & \mathbf{b}\cdot\mathbf{y} \\ \text{primal} & & A\mathbf{x}\le\mathbf{b} & \text{dual} & & A^T\mathbf{y}\ge\mathbf{c}\,. \\ & & \mathbf{x}\ge\mathbf{0} & & & \mathbf{y}\ge\mathbf{0} \end{array}$$

Then $\mathbf{c}\cdot\mathbf{x}^* \le \mathbf{b}\cdot\mathbf{y}^*$.

**Proof:** We note that $A^T\mathbf{y}\ge\mathbf{c}$ and $\mathbf{x}\ge\mathbf{0}$ yields $\mathbf{x}^T A^T\mathbf{y}\ge\mathbf{x}^T\mathbf{c}$ which we write as $\mathbf{c}\cdot\mathbf{x}=\mathbf{x}^T\mathbf{c}\le\mathbf{x}^T A^T\mathbf{y}$. Similarly $A\mathbf{x}\le\mathbf{b}$ and $\mathbf{y}\ge\mathbf{0}$ yields $\mathbf{y}^T A\mathbf{x}\le\mathbf{y}^T\mathbf{b}$ which we write as $\mathbf{y}^T A\mathbf{x}\le\mathbf{y}^T\mathbf{b}=\mathbf{b}\cdot\mathbf{y}$. Now $\mathbf{x}^T A^T\mathbf{y}$ is a $1\times 1$ matrix and so $\mathbf{x}^T A^T\mathbf{y}=\left(\mathbf{x}^T A^T\mathbf{y}\right)^T=\mathbf{y}^T\left(A^T\right)^T(\left(\mathbf{x}^T\right)^T=\mathbf{y}^T A\mathbf{x}$. We obtain

$$\mathbf{c}\cdot\mathbf{x}=\mathbf{x}^T\mathbf{c}\le\mathbf{x}^T A^T\mathbf{y}=\mathbf{y}^T A\mathbf{x}\le\mathbf{y}^T\mathbf{b}=\mathbf{b}\cdot\mathbf{y}.$$

We read off $\mathbf{c}\cdot\mathbf{x}\le\mathbf{b}\cdot\mathbf{y}$. ∎

The case of equality is of course of great interest and Strong Duality and Complementary Slackness deal with equality. Nonetheless, Weak Duality is of independent interest and is a model for other optimization problems for which we have no Strong Duality.