Linear Programs ask to minimize or maximize a linear function subject to linear inequalities either "=" or "≤" or "≥". Strict inequalities "<" or ">" are not allowed. Any Linear Programming problem can be put in standard inequality form which is the maximization of a linear objective function subject to inequalities each of which is a linear function of the variables less than or equal to some constant and in addition to the requirement that each variable is positive.

$$
\begin{array}{llllllll}
\max & c_1x_1 & +c_2x_2 & +\cdots & + & +c_nx_n & (= & z) \quad \textit{(objective function)} \\
 & a_{11}x_1 & +a_{12}x_2 & +\cdots & + & +a_{1n}x_n & \le & b_1 \quad \textit{(constraints)} \\
 & a_{21}x_1 & +a_{22}x_2 & +\cdots & + & +a_{2n}x_n & \le & b_2 \\
 & & & \vdots & & & & \vdots \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots & + & +a_{mn}x_n & \le & b_m \\
 & & x_1, x_2, \ldots, x_n \ge 0 & & & & & \quad \textit{(positivity constraints)}
\end{array}
$$

Now this is best manipulated in matrix and vector notation but when the notation ever becomes unfamiliar to you, you should revert to the original data in the form above.

$$\mathbf{c} = (c_1, c_2, \ldots, c_n)^T, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T,$$

$$\mathbf{b} = (b_1, b_2, \ldots, b_m)^T, \mathbf{0} = (0, 0, \ldots, 0)^T.$$

Note that we have not given a specific length to $\mathbf{0}$ and that it is because the length that was intended will be clear from context. For example $\mathbf{0} \cdot \mathbf{x}$ would imply that we are discussing a 0-vector of n entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Our standard inequality form can now be written in matrix notation:

$$
\begin{array}{lccc}
\max & \mathbf{c} \cdot \mathbf{x} & & \\
 & A\mathbf{x} & \le & \mathbf{b}\,. \\
 & \mathbf{x} & \ge & \mathbf{0}
\end{array}
$$

Note that we cannot directly deal with strict inequalities. LINDO for example will not let you enter them although the input format using < might suggest otherwise until you realize that LINDO interprets < as 'less than or equals'. Here are the various transformations required to take a Linear Program into standard inequality form.

CASE 1. min used in objective function instead of max.

If we have an objective function min $\mathbf{c} \cdot \mathbf{x}$ then we can replace the objective function by max $(-\mathbf{c}) \cdot \mathbf{x}$ and the new LP will have the property that an optimal solution to the original LP will still be optimal for the new transformed LP but of course the value for the objective function will be the negative.

CASE 2. The objective function is max $\mathbf{c} \cdot \mathbf{x} +$ constant.

In this case, we can simply delete the constant and the new LP will have the property that an optimal solution (for the decision variables) to the original LP will still be optimal for the new transformed LP but of course the value for the objective function will reduced by the deleted constant.

CASE 3. $\geq$ inequality instead of an $\leq$ inequality.

An inequality $\mathbf{a} \cdot \mathbf{x} \geq b$ is equivalent to the inequality $(-\mathbf{a}) \cdot \mathbf{x} \leq (-b)$.

CASE 4. An equality instead of an $\leq$ inequality.

An equality $\mathbf{a} \cdot \mathbf{x} = b$ is equivalent to two inequalities $\mathbf{a} \cdot \mathbf{x} \leq b$ and $(-\mathbf{a}) \cdot \mathbf{x} \leq (-b)$.

CASE 5. The variable x_i is not subject to inequality $x_i \geq 0$ but is subject to either $x_i \leq u_i$ or $x_i \geq l_i$.

If we have the inequality $x_i \geq l_i$ which is equivalent to $x_i - l_i \geq 0$, then we can do a linear shift on the decision variable using $x_i' = x_i - l_i$ (and $x_i = x_i' + l_i$) and eliminate x_i while replacing it by x_i' where now $x_i' \geq 0$. Obviously when you are done you are interested in determining the values of the original decision variable and so would again use $x_i = x_i' + l_i$ to do so. Similarly if we have the inequality $x_i \leq u_i$ which is equivalent to $u_i - x_i \geq 0$, then we can do a linear transformation on the decision variable using $x_i' = u_i - x_i$ (i.e. $x_i = u_i - x_i'$) and eliminate x_i while replacing it by x_i' with $x_i' \geq 0$.

CASE 6. The variable x_i is a free variable (no inequalities of form $x_i \leq u_i$ or $x_i \geq l_i$).

This case is handled by a more unexpected transformation. Create two new positive variables x_i', x_i'' and then set $x_i = x_i' - x_i''$ with $x_i', x_i'' \geq 0$. Thus we eliminate x_i and replace it by two new variables and so when we have found an optimal solution we then use $x_i = x_i' - x_i''$ again to determine x_i. As it stands, the substitution $x_i = x_i' - x_i''$ does not yield unique values for the new variables x_i', x_i''. You will discover that the simplex method does force the following unique assignment:

$$x_i' = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad x_i'' = \begin{cases} -x_i & \text{if } x_i \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

A few examples of these six cases are in the practice for the first quiz.

If there is one skill you must remember is that *multiplying an inequality by -1 changes the direction of the inequality.*

One can discuss a Standard Equality Form, which would look like

$$\begin{array}{rrcl} \max & \mathbf{c} \cdot \mathbf{x} & & \\ & A\mathbf{x} & = & \mathbf{b}. \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

Note that the standard simplex method we describe does this but that our duality theory is stated for Standard Inequality Form.