One of the interesting features of MATH 340 is that it is dominated by an algorithm. In fact we prove the important Duality theorems using our algorithm. This approach of proving a theorem using an algorithm may be unusual to you. For those who have taken some computer science courses, you will be alert to the need to check boundary cases and make sure the algorithm will always terminate in finite time.

We have set up the problem as follows

Initializations

Start with an LP in standard inequality form.

Add slacks and form the initial dictionary. (A dictionary expresses a chosen set of basic variables corresponding to a column basis of $[A|I]$ in terms of the non basic variables).

If the dictionary has an associated basic feasible solution then go to Phase Two. Otherwise commence Phase One to find a basis yielding a basic feasible solution.

Phase One.

Add x_0 to the right side of each of the equations of the initial dictionary. Introduce an objective function $w = -x_0$.

Do a **Non-standard pivot to feasibility**, so called non-standard because the pivot rules are definitely non-standard and a pivot to feasibility because the new dictionary, with x_0 in the basis, yields a basic feasible solution.

Now pivot to maximize w. Our standard pivot rules preserve feasibility. With Bland's rule we can ensure no basis is repeated and hence the algorithm must terminate in a finite number of pivots. Thus the algorithm will either terminate with no entering variable or with no leaving variable.

If we are unable to find a entering variable it will be because the cuurent basic feasible solution is an optimal solution maximizing w. If we terminate with a dictionary whose basic feasible solution has $x_0 \neq 0$ and hence, since $x_0 \geq 0$, with $w < 0$, then we deduce that there is no feasible solution. We stop the simplex algorithm. If we terminate with a dictionary whose basic feasible solution has $x_0 = 0$ and hence with $w = 0$, then we deduce that x_0 is not a basic variable (By Anstee's rules, when x_0 is driven to 0 it will leave the basis in preference to any other variable that is driven to 0 since 0 is the smallest subscript). We may delete x_0 from our dictionary to obtain a dictionary consisting of the original variables and slacks where the associated basis solution is feasible.

If we are unable to find a leaving variable it will be because the LP is unbounded. But $w = -x_0 \leq 0$ so this case cannot occur.

Phase Two.

We reintroduce z, the objective function of our original LP. Using our dictionary equations if necessary we re-express z in terms of non-basic variables

Now pivot to maximize z. Our standard pivot rules preserve feasibility. With Bland's rule we can ensure no basis is repeated and hence the algorithm must terminate in a finite number of pivots. Thus the algorithm will either terminate with no entering variable or with no leaving variable.

If we are unable to find a entering variable it will be because the current basic feasible solution is an optimal solution for the LP. We stop the simplex algorithm.

If we are unable to find a leaving variable it will be because the LP is unbounded. We obtain a parametric set of feasible solutions to the LP with the parameter t equal to the entering variable and for which the value of the objective tends to ∞ as the parameter goes to ∞. We stop the simplex algorithm.

I will now give an example of the two phase method that we did in class. You can also witness examples of the two phase method in the practice for quiz2.

$$\begin{array}{lrrrrl} \text{Maximize} & -x_1 & +3x_2 & +x_3 & +x_4 & \\ & 2x_1 & +x_2 & & -x_4 & \le 4 \\ & -2x_1 & & +x_3 & +x_4 & \le -2 \\ & & 2x_2 & +2x_3 & & \le 3 \end{array} \qquad x_1, x_2, x_3, x_4 \ge 0$$

Solution:

Phase One: We might try to write

$$\begin{array}{rclrrrrr} x_5 & = & 4 & -2x_1 & -x_2 & & -x_4 \\ x_6 & = & -2 & +2x_1 & & -x_3 & -x_4 \\ x_7 & = & 3 & & -2x_2 & -2x_3 & \\ z & = & & -x_1 & +3x_2 & +x_3 & +x_4 \end{array}$$

But the associated 'obvious' solution (the basic solution associated with dictionary) has $x_6 = -2$ so it is not feasible. While we could pivot it would be hopeless to describe the pivot rules for this. Recall we choose the leaving variable to preserve feasibility but that won't work if you start with an infeasible solution.

Instead we add in the artificial variable x_0 to make this work:

$$\begin{array}{rclrrrrr} x_5 & = & 4 & -2x_1 & -x_2 & & -x_4 & +x_0 \\ x_6 & = & -2 & +2x_1 & & -x_3 & -x_4 & +x_0 \\ x_7 & = & 3 & & -2x_2 & -2x_3 & & +x_0 \\ w & = & & & & & & -x_0 \end{array}$$

The new objective function is $\max -x_0$ and so it attempts to drive x_0 to 0. We can take x_0 intially large to find a feasible solution

$x_5 = 4 + x_0 \ge 0$ so $x_0 \ge -4$

$x_6 = -2 + x_0 \ge 0$ so $x_0 \ge 2$

$x_7 = 3 + x_0 \ge 0$ so $x_0 \ge -3$.

We conclude that $x_0 = 2$ drives x_6 to 0 while having $x_5 \ge 0$ and $x_7 \ge 0$. These are not the usual rules and so we call this the *special pivot to feasibility.*

x_0 enters and x_6 leaves (Specal pivot to feasibility)

$$\begin{array}{rclrrrrr} x_5 & = & 6 & -4x_1 & -x_2 & +x_3 & +2x_4 & +x_6 \\ x_0 & = & 2 & -2x_1 & & +x_3 & +x_4 & +x_6 \\ x_7 & = & 5 & -2x_1 & -2x_2 & -x_3 & +x_4 & +x_6 \\ w & = & -2 & +2x_1 & & -x_3 & -x_4 & -x_6 \end{array}$$

This is a traditional dictionary and we now attempt to pivot to drive x_0 to 0 at which point we will delete it. x_1 enters and x_0 leaves (that was easy!!).

$$\begin{array}{rclrrrrr} x_5 & = & 2 & +2x_0 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & -\frac{1}{2}x_0 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & +x_0 & -2x_2 & -2x_3 & & \\ w & = & & -x_0 & & & & \end{array}$$

We may now delete x_0 and w:

$$\begin{array}{rclrrrr} x_5 & = & 2 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & -2x_2 & -2x_3 & & \end{array}$$

This finished Phase one. We must reintroduce $z = -x_1 + 3x_2 + x_3 + x_4$ to begin Phase two but in the form that it is written in terms of the non basic variables x_2, x_3, x_4, x_6. To do so we need only substitute for x_1 in this simple case

$$z = -(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6) + 3x_2 + x_3 + x_4 = -1 + 3x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6$$

yieldiing the dictionary

$$\begin{array}{ccccccc} x_5 & = & 2 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & -2x_2 & -2x_3 & & \\ z & = & -1 & +3x_2 & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & -\frac{1}{2}x_6 \end{array}$$

We now apply our standard pivot rules again and choose x_2 to enter and have x_7 leave:

$$\begin{array}{ccccccc} x_5 & = & \frac{1}{2} & +\frac{1}{2}x_7 & & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_2 & = & \frac{3}{2} & -\frac{1}{2}x_7 & -\frac{1}{2}x_3 & & \\ z & = & \frac{7}{2} & -\frac{3}{2}x_7 & -\frac{5}{2}x_3 & +\frac{1}{2}x_4 & -\frac{1}{2}x_6 \end{array}$$

Note that the annoying fractions are rarely asked in test questions and not in Quiz 2. We apply our pivot rules and have x_4 enter but find no leaving variable and we are in the unbounded case with parametric solutions $x_4 = t$, $x_7 = x_3 = x_6 = 0$ and $x_5 = \frac{1}{2}$, $x_1 = 1 + \frac{1}{2}t$ and $x_2 = \frac{3}{2}$ with $z = \frac{7}{2} + \frac{1}{2}t$ showing that the LP is unbounded. We take $t \geq 0$ and $t \to \infty$.

Note that each Phase looks like a traitional simplex method except for the special pivot to feasibility that starts Phase one. If you fail to do this correctly, you will have infeasible dictionaries (negative values for basic variables when non basics are set to 0) and you should stop there and try to find the correct pivot choice.

Please check to practice for quiz 2 to see more examples.