MATH 340 Example of Infeasibility. Richard Anstee

We give an example of an LP where we are unable to drive the artificial variable x_0 to 0. Or equivalently we find that when we maximize $w = -x_0$ we are ubable to get ot to zero so we cannot proceed to Phase two. I've neglected to give the problem with an objective function since it is not used in what follows.

We considered the following set of inequalities.

$$\begin{array}{rrcr} x_1 & +x_2 & \leq & 1 \\ -x_1 & -2x_2 & \leq & -3 \end{array} \quad x_1, x_2 \geq 0$$

We add slack variables x_3, x_4 and the artificial variable x_0.

$$\begin{array}{rcrrrr} x_3 & = & 1 & -x_1 & -x_2 & +x_0 \\ x_4 & = & -3 & x_1 & +2x_2 & +x_0 \\ w & = & & & & -x_0 \end{array}$$

We choose x_0 to enter and x_4 leaves in our non standard pivot to feasibility.

$$\begin{array}{rcrrrr} x_3 & = & 4 & -2x_1 & -3x_2 & +x_4 \\ x_0 & = & 3 & -x_1 & -2x_2 & +x_4 \\ w & = & -3 & +x_1 & +2x_2 & -x_4 \end{array}$$

We choose x_2 to enter and x_3 to leave.

$$\begin{array}{rcrrrr} x_2 & = & \frac{4}{3} & -\frac{2}{3}x_1 & -\frac{1}{3}x_3 & +\frac{1}{3}x_4 \\ x_0 & = & \frac{1}{3} & +\frac{1}{3}x_1 & +\frac{2}{3}x_3 & +\frac{1}{3}x_4 \\ w & = & -\frac{1}{3} & -\frac{1}{3}x_1 & -\frac{2}{3}x_3 & -\frac{1}{3}x_4 \end{array}$$

This yields the optimal solution if we are trying to maximize $w = -x_0$ and so we deduce there are no feasible choices for x_1, x_2 to satisy the two inequalities. This would be obvious if you graphed the two inequalities.

Now there are *Magic Coefficients* in the final w row. Take the coefficients of the two slack variables x_3, x_4 in the w row. They are $-\frac{2}{3}$ and $-\frac{1}{3}$. Now x_3 is the slack for the inequality $x_1 + x_2 \leq 1$ and x_4 is the slack for the inequality $-x_1 - 2x_2 \leq -3$. Now

$$\frac{2}{3}(x_1 + x_2) \leq \frac{2}{3} \cdot 1$$

$$\text{so} \quad \frac{2}{3}x_1 + \frac{2}{3}x_2 \leq \frac{2}{3}$$

using the properties of multiplying an inequality by a positive constant. Similarly

$$\frac{1}{3}(-x_1 - 2x_2) \leq \frac{1}{3} \cdot -3$$

$$\text{so} \quad -\frac{1}{3}x_1 - \frac{2}{3}x_2 \leq -1$$

Adding these together yields

$$\frac{1}{3}x_1 \leq -\frac{1}{3},$$

which is impossible (given $x_1 \geq 0$). So the two inequalities are infeasible.

We will explain the *Magic* in later work.