Math 340 **The game of Morra and a variation** Richard Anstee

This game is well described in our text by Chvátal. Our course website has the LINDO input file. The game has easy but sometimes confusing rules about payoffs. Obviously the payoff matrix will be given correctly.

Each player hides one or two francs and then tries to guess how many francs the other has hidden. If neither is correct or bot are correct, then the payoff is 0. If one is correct, then the correct player gets a payoff equal to the total hidden. This is a bit tricky. You might expect to take the francs from the other player but that is not the rule. Best to think of the hidden coins as tokens and the payoff is in some other currency!

The game is set up in the text as a game between Trucula (the row player) and Claude (the column player). I would certainly guess that Claude refers to Claude Berge, on of the famous graph theorists of the last century. But Trucula??

The payoff matrix is as below. A strategy $[i, j]$ refers to hiding i francs and guessing opponent has hidden j francs. As usual, the payoffs are the payoffs to the row player, in this case Trucula.

$$\begin{array}{cc} & \text{Claude strategies} \\ & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \text{Trucula strategies} \begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} & \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix} \end{array}$$

Note that this matrix is skew symmetric ($A = -A^T$) and so the game is the same for both players and so the strategies are the same for both players. The LP to solve the game for Trucula is

$$\begin{array}{llllll} \max & z & & & & \\ & z & & +2x_2 & -3x_3 & & \leq 0 \\ & z & -2x_1 & & & +3x_4 & \leq 0 \\ & z & +3x_1 & & & -4x_4 & \leq 0 \\ & z & & -3x_2 & +4x_3 & & \leq 0 \\ & & x_1 & +x_2 & +x_3 & +x_4 & = 1 \end{array} \quad \text{free } z$$

The dual is

$$\begin{array}{llllll} \min & w & & & & \\ & w & & -2y_2 & +3y_3 & & \geq 0 \\ & w & +2y_1 & & & -3y_4 & \geq 0 \\ & w & -3y_1 & & & +4y_4 & \geq 0 \\ & w & & +3y_2 & -4y_3 & & \geq 0 \\ & & y_1 & +y_2 & +y_3 & +y_4 & = 1 \end{array} \quad \text{free } z$$

We obtain a solution for the row player Trucula $\mathbf{x} = (0, 3/5, 2/5, 0)^T$ with $z = 0$ which will of course also work for the column player Claude. We can now employ question 3 from assignment 3. Note that $w = 0$ at optimality in the dual using Strong Duality. Then by complementary slackness:

$$\begin{array}{lcl} x_2 > 0 & \Longrightarrow & (\text{by C.S. }) + 2y_1 - 3y_4 = 0 \\ x_3 > 0 & \Longrightarrow & (\text{by C.S. }) - 3y_1 + 4y_4 = 0 \\ -3x_2 + 4x_3 < 0 & \Longrightarrow & (\text{by C.S. })y_1 = 0 \end{array}$$

This yields $y_1 = y_4 = 0$. But now we use feasibility in dual, namely $-2y_2+3y_3 \geq 0$ and $3y_2-4y_3 \geq 0$. In general solving some inequalities is much like solving for all feasible solutions of an LP and so might be hopeless. In this case we can do it. We have $y_2+y_3 = 1$ and so we can substitute $y_3 = 1-y_2$. Our inequality $-2y_2 + 3y_3 \geq 0$ becomes $-2y_2 + 3(1-y_2) \geq 0$ and so $3 \geq 5y_2$ so that $y_2 \leq 3/5$. Our inequality $3y_2-4y_3 \geq 0$ becomes $3y_2-4(1-y_2) \geq 0$ and so $7y_2 \geq 4$ so that $y_2 \geq 4/7$. This gives us all solutions. In particular we also get the solutions $\mathbf{y} = (0, 3/5, 2/5, 0)^T$ and $\mathbf{y} = (0, 4/7, 3/7, 0)^T$ and hence we get all convex combinations, namely $\mathbf{y} = (0, \lambda(3/5)+(1-\lambda)(4/7), \lambda(2/5)+(1-\lambda)(3/7), 0)^T$ where $\lambda \in [0, 1]$. You can check $A\mathbf{y}$ or indeed $\mathbf{x}^T A$ the value of the game is 0.

Chv'atal suggests an interesting variation to Morra where the coins are hidden first but then Claude guesses before Trucula. Thus Trucula has a little bit more information which can maybe used to Trucula's advantage. Create new strategies for Trucula

$[1, S]$	hide 1 and guess same as Claude
$[1, D]$	hide 1 and guess differently from Claude
$[2, S]$	hide 2 and guess same as Claude
$[2, D]$	hide 2 and guess differently from Claude

We get a new payoff matrix

$$\text{Trucula strategies} \quad \begin{array}{c} \\ {[1,1]} \\ {[1,2]} \\ {[2,1]} \\ {[2,2]} \\ {[1,S]} \\ {[1,D]} \\ {[2,S]} \\ {[2,D]} \end{array} \begin{array}{c} \text{Claude strategies} \\ \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \left[\begin{array}{cccc} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -3 & 3 \\ -2 & 2 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 4 & -4 \end{array} \right] \end{array}$$

When we solve this game Trucula has an optimal strategy with an expected payoff of 4/99 using the strategy

$\mathbf{x} = (0, 55/99, 40/99, 0, 0, 2/99, 0, 1/99)^T$

and Claude will use

$\mathbf{y} = (28/99, 30/99, 21/99, 20/99)^T$

You can verify (if you like fractions) that $\mathbf{x}^T A$ is in fact a 1×4 vector with all entries 4/99.

Now you can see how the computational aspects of Game Theory can help you greatly.