MATH 340 Example of Degeneracy in pivoting process. Richard Anstee

We give an example of an LP where we are forced to do what we call *degenerate pivots*. We still obtain an optimal solution(s). Degenerate pivots can result in cycling and you should read Chvatal's example.

We considered the following LP in standard inequality form

$$\begin{array}{lrrrcl} \max & 2x_1 & +2x_2 & +x_3 & & \\ & 2x_1 & & +x_3 & \leq & 4 \\ & x_1 & +x_2 & & \leq & 1 \\ & x_1 & & +x_3 & \leq & 1 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

We add slack variables x_4, x_5, x_6 corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. We form our first dictionary

$$\begin{array}{rclrrr} x_4 & = & 4 & -2x_1 & & -x_3 \\ x_5 & = & 1 & -x_1 & -x_2 & \\ x_6 & = & 1 & -x_1 & & -x_3 \\ z & = & & 2x_1 & +2x_2 & +x_3 \end{array}$$

It is traditional to use z for the objective function. There is an *obvious solution* to these 4 equations, namely $x_4 = 4$, $x_5 = 1$, $x_6 = 1$ and $x_1 = x_2 = x_3 = 0$ with $z = 0$. (this is called a *basic feasible solution*)

We now use Anstee's rule trying to increase a variable from 0 in the current *obvious solution* so we greedily choose x_1 to increase and hence *enter*. We leave $x_2 = x_3 = x_4 = 0$. The choice of x_1 as the variable with the largest coefficient in dictionary expression for z (and in the case of ties choosing the variable of smallest subscript) is called **Anstee's Rule** in this course. If there is a tie we choose the variable with the smallest subscript.

$$\begin{array}{rclr} x_4 & = & 4 & -2x_1 \\ x_5 & = & 1 & -x_1 \\ x_6 & = & 1 & -x_1 \\ z & = & & 4x_1 \end{array}$$

We deduce that x_1 can be increased to 1 while decreasing x_5 and x_6 to 0. We obtain a new dictionary by having x_1 only appear on the left and x_5 is now on the right of the equation signs. We choose x_5 to leave by Anstee's rule since there is a tie for the leaving variable and we choose the one with the smallest subscript.

$$\begin{array}{rclrrr} x_4 & = & 2 & +2x_5 & +2x_2 & -x_3 \\ x_1 & = & 1 & -x_5 & -x_2 & \\ x_6 & = & 0 & +x_5 & +x_2 & -x_3 \\ z & = & 2 & -2x_5 & & +x_3 \end{array}$$

This dictionary yields a basic solution $x_4 = 2$, $x_1 = 1$, $x_6 = 0$ and $x_5 = x_2 = x_3 = 0$ with $z = 2$. One of our basic variable, x_6, is 0. Such a basic solution is called a *degenerate basic solution*.

By Anstee's rule we would wish to increase x_3 leaving $x_5 = x_2 = x_3 = 0$. We decide we can increase x_3 from 0 to 0 while driving x_6 from 0 down to 0. Perhaps this is a bit difficult to understand but this is the wording we want to use to mimic our more standard pivots. We call this a *degenerate pivot* because no variable will change value in the basic solution. The new dictionary with x_3 entering and x_6 leaving is:

$$\begin{array}{rcrrrr} x_4 & = & 2 & +x_5 & +x_2 & +x_6 \\ x_1 & = & 1 & -x_5 & -x_2 & \\ x_3 & = & 0 & +x_5 & +x_2 & -x_6 \\ z & = & 2 & -x_5 & +x_2 & -x_6 \end{array}$$

Again, the current basic solution is degenerate (since $x_3 = 0$) and the basic solution is unchanged but note that the basis has changed which is significant. We now have x_2 enter and then x_1 leaves in a non-degenerate pivot (the basic solution changes). Of course we would be equally happy if the pivot was degenerate but resulted in a dictionary that reveals the basic solution is optimal.

$$\begin{array}{rcrrrr} x_4 & = & 3 & & -x_1 & +x_6 \\ x_2 & = & 1 & -x_5 & -x_1 & \\ x_3 & = & 1 & & -x_1 & -x_6 \\ z & = & 3 & -2x_5 & -x_1 & -x_6 \end{array}$$

The basic solution is now $x_4 = 3$, $x_2 = 1$ and $x_3 = 1$ with $x_5 = x_1 = x_6 = 0$ and $z = 3$. We see this is an optimal solution. This example has us do a degenerate pivot but somehow, from the new basis $\{x_1, x_1.x_3\}$, we can now see a beneficial pivot (with z increasing).

Degeneracy is relatively common in LP's and you proceed as we have described. It is possible to enter a cycle (one example is posted) but this is extremely rare in practice; it essentially never happens so in practice you would not dwell on the possibility. But in order to show the termination of the Simplex algorithm, we need to have a way to avoid cycling.