MATH 340 Another Two Phase example. Richard Anstee

I will now give an example of the two phase method that we did in class. You can also witness examples of the two phase method in the practice for quiz2.

$$\begin{array}{lrrrl}
\text{Maximize} & -2x_1 & -x_2 & -x_3 & \\
& -x_1 & & -x_3 & \leq -1 \\
& -x_1 & -x_2 & & \leq -2 \\
& & -x_2 & +x_3 & \leq -1
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

Solution:

Phase One: We might try to write

$$\begin{array}{rcrrrr}
x_4 & = & -1 & +x_1 & & +x_3 \\
x_5 & = & -2 & +x_1 & +x_2 & \\
x_6 & = & -1 & & +x_2 & -x_3 \\
z & = & & -2x_1 & -x_2 & -x_3
\end{array}$$

But the associated 'obvious' solution (the basic solution associated with dictionary) has $x_4 = -1$, $x_5 = -2$ and $x_6 = -1$ so it is not feasible. While we could pivot it would be hopeless to describe the pivot rules for this. Recall we choose the leaving variable to preserve feasibility but that won't work if you start with an infeasible solution.

Instead we add in the artificial variable x_0 to make this work:

$$\begin{array}{rcrrrrr}
x_4 & = & -1 & +x_1 & & +x_3 & +x_0 \\
x_5 & = & -2 & +x_1 & +x_2 & & +x_0 \\
x_6 & = & -1 & & +x_2 & -x_3 & +x_0 \\
w & = & & & & & -x_0
\end{array}$$

The new objective function is max $-x_0$ and so it attempts to drive x_0 to 0. We can take x_0 intially large to find a feasible solution

$x_4 = -1 + x_0 \geq 0$ so $x_0 \geq 1$

$x_5 = -2 + x_0 \geq 0$ so $x_0 \geq 2$

$x_6 = -1 + x_0 \geq 0$ so $x_0 \geq 1$.

We conclude that $x_0 = 2$ drives x_5 to 0 while having $x_4 = 1 \geq 0$ and $x_6 = 1 \geq 0$. So we choose x_5 to leave (it is the last variable driven to 0 as we increase x_0). These are not the usual rules and so we call this the *special pivot to feasibility.*

x_0 enters and x_5 leaves (Specal pivot to feasibility)

$$\begin{array}{rcrrrrr}
x_4 & = & 1 & & -x_2 & +x_3 & +x_5 \\
x_0 & = & 2 & -x_1 & -x_2 & & +x_5 \\
x_6 & = & 1 & -x_1 & & -x_3 & +x_5 \\
w & = & -2 & +x_1 & +x_2 & & -x_5
\end{array}$$

This is a traditional dictionary and we now attempt to pivot to drive x_0 to 0 at which point we will delete it. There is a tie for the entering variable between x_1 and x_2. **Anstee's Rule** asks you to choose the smallest subscript in the event of ties. So x_1 enters and x_6 leaves.

$$\begin{array}{rcrrrrr}
x_4 & = & 1 & & -x_2 & +x_3 & +x_5 \\
x_0 & = & 1 & +x_6 & -x_2 & +x_3 & \\
x_1 & = & 1 & -x_6 & & -x_3 & +x_5 \\
w & = & -1 & -x_6 & +x_2 & -x_3 &
\end{array}$$

We have made progress (x_0 is now just 1) but carry on. x_2 enters. There is a tie for the leaving variable between x_0 and x_4. **Anstee's Rule** asks you to choose the smallest subscript in the event of ties. The choice x_0 now becomes clear. When x_0 is driven to 0 in the pivoting process it will leave the basis. So x_2 enters and x_0 leaves.

$$\begin{array}{rcrrrrr} x_4 & = & 0 & -x_6 & +x_0 & & +x_5 \\ x_2 & = & 1 & +x_6 & -x_0 & +x_3 & \\ x_1 & = & 1 & -x_6 & & -x_3 & +x_5 \\ w & = & 0 & & -x_0 & & \end{array}$$

We may now delete x_0 and w (note that we want x_0 to be on the left side, one of the non basic variables):

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 1 & +x_6 & +x_3 & \\ x_1 & = & 1 & -x_6 & -x_3 & +x_5 \end{array}$$

This finishes Phase one. Two pivots and we have driven x_0 to 0 and so have a feasible dictionary to begin Phase Two. We must reintroduce $z = -2x_1 - x_2 - x_3$ to begin Phase Two but in the form that it is written in terms of the non basic variables x_3, x_5, x_6. To do so we need substitute for x_1 and x_2 in this simple case

$$z = -2(1 - x_6 - x_3 + x_5) - (1 + x_6 + x_3) - x_3 = -3 + x_6 - 2x_5$$

yielding the dictionary

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 1 & +x_6 & +x_3 & \\ x_1 & = & 1 & -x_6 & -x_3 & +x_5 \\ z & = & -3 & +x_6 & & -2x_5 \end{array}$$

We now apply our standard pivot rules again and choose x_6 to enter and have x_4 leave. Well this is not entirely what we are used to doing. We are increasing x_6 from 0 to 0 while driving x_4 from 0 down to 0. This is called a degenerate pivot. We obtain the following dictionary.

$$\begin{array}{rcrrrr} x_6 & = & 0 & -x_4 & & +x_5 \\ x_2 & = & 1 & -x_4 & +x_3 & +x_5 \\ x_1 & = & 1 & +x_4 & -x_3 & \\ z & = & -3 & -x_4 & & -x_5 \end{array}$$

Now the actual basic solution has not changed. It is (1,1,0,0,0,0) with $z = -3$ in both cases. But the set of basic variables has changed and moreover it is now clear that we are at optimality. This sort of thing occurs in Mathematics, to see an optimal solution you often have to transform the problem.