1. (for practice only; won't be graded but solution will be posted) Consider the LP:

$$\begin{array}{lrrrl} \text{Maximize} & -x_1 & -x_2 & +x_3 & \\ & -x_1 & & -x_3 & \le -1 \\ & -x_1 & -x_2 & & \le -2 \\ & & -x_2 & +x_3 & \le -1 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

The final dictionary is

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 2 & & -x_1 & +x_5 \\ x_3 & = & 1 & -x_6 & -x_1 & +x_5 \\ z & = & -1 & -x_6 & -x_1 & \end{array}$$

We now consider adding the constraint $x_1 + x_2 + x_3 \le 1$. Use the method of class and use our dual simplex method to solve. You will find the dual is unbounded (and hence the primal is infeasible). Give a parametric set of feasible solutions to the dual whose objective function goes to $-\infty$.

2. Consider our standard LP: $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$. Assume $\mathbf{u}$ and $\mathbf{v}$ are both feasible solutions to the LP. Show that for any choice $\lambda \in [0,1]$, that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is also a feasible solution to the LP. You could start with $\lambda = 1/2 = (1-\lambda)$ to try on this problem. Also show that if $\mathbf{u}$ and $\mathbf{v}$ are both optimal solutions to the LP , then $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ are optimal solutions for any choice $\lambda \in [0,1]$.

   Note: A set of points P in $\mathbf{R}^n$ is called *convex* if for every pair $\mathbf{x}, \mathbf{y} \in P$, that all the points on the *line segment* joining them is also in P. The set of points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ is $\{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \,:\, \lambda \in [0,1]\} = \{\mathbf{y} + \lambda(\mathbf{x} - \mathbf{y}) \,:\, \lambda \in [0,1]\}$.

3. We know by our Marginal Value Theorem that the marginal values given by the dual variables predict the exact changes in the objective function for changes $\Delta\mathbf{b}$ to $\mathbf{b}$ for which $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \ge \mathbf{0}$. Assume we know (perhaps from LINDO output) that $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \ge \mathbf{0}$ for $\Delta\mathbf{b} = (6,0,0,0)^T$ and also for $\Delta\mathbf{b} = (0,0,8,0)^T$. Show that $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \ge \mathbf{0}$ for $\Delta\mathbf{b} = (3,0,4,0)^T$. You might note that $\frac{1}{2} \times (6,0,0,0)^T + \frac{1}{2} \times (0,0,8,0)^T = (3,0,4,0)^T$.

4.

a) Consider the game given by payoff matrix A below (the payoff to the row player).

$$A = \begin{bmatrix} 5 & -3 & -4 \\ -4 & -2 & 5 \end{bmatrix}$$

State explictly the LP for the row player. Deduce a bound on $v(A)$ if we use the mixed strategy for the column player $\mathbf{y} = (1/4, 1/2, 1/4)^T$.

b) Find the optimal strategy for the row player and the column player for the game whose payoff matrix (for the row player) is as follows with $e, f > 0$. Hint: Try a column player strategy $(1/2, 1/2)^T$.

$$A = \begin{bmatrix} e & -e \\ -f & f \end{bmatrix}$$

5. (from an old exam) We have a gasoline blending problem where we may mix four gasoline products x_1, x_2, x_3, x_4 from 3 types of gas (gas1,gas2,gas3). The availability of the three gases is given as a function of a parameter p:

$$\begin{array}{lccccll} \text{Maximize} & 12.1x_1 & +8.3x_2 & +14.2x_3 & 18.1x_4 & (=z) & \\ (gas1) & 2x_1 & +x_2 & +3x_3 & +6x_4 & \leq 37+2p & \\ (gas2) & 3x_1 & & +2x_3 & +x_4 & \leq 42+p & x_1, x_2, x_3, x_4 \quad \geq \quad 0 \\ (gas3) & x_1 & +4x_2 & +2x_3 & +x_4 & \leq 39-3p & \end{array}$$

We are interested in the value of the objective function z as a function of the parameter p near $p = 1$. We make p a variable (and move it to the other side of the inequalities), and make p free (allowing it to be negative). We also impose the somewhat arbitrary condition $p \leq 1$ and then use this constraint in our analysis. We send this off to LINDO placing the variables on the left. The LINDO output will be useful for parts b),c).

a) What happens to z in our original LP when $p = \frac{-37}{2}$? What happens for $p < \frac{-37}{2}$? Explain.

b) What is z for $p = 1$? What is the slope of the graph of z as a function of p at that point? For what range on p is the slope valid.

c) Explain the interesting coincidence that:
$2(2.525000) + 1(1.868750) - 3(1.443750) = 2.587500$.

The input to LINDO is:

```
max 12.1x1 + 8.3x2 + 14.2x3 + 18.1x4
subject to
gas1) 2x1+x2+3x3+6x4-2p<37
gas2) 3x1+2x3+x4-p<42
gas3) x1+4x2+2x3+x4+3p<39
param)p<1
end
free p
```

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

1) 230.8062

VARIABLE	VALUE	REDUCED COST
$X1$	11.875000	0.000000
$X2$	4.187500	0.000000
$X3$	3.687500	0.000000
$X4$	0.000000	0.362500
P	1.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
$GAS1)$	0.000000	2.525000
$GAS2)$	0.000000	1.868750
$GAS3)$	0.000000	1.443750
$PARAM)$	0.000000	2.587500

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$X1$	12.100000	0.161111	2.990000
$X2$	8.300000	0.322222	4.620000
$X3$	14.200000	4.271429	0.093548
$X4$	18.100000	0.362500	$INFINITY$
P	0.000000	$INFINITY$	2.587500

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$GAS1$	37.000000	16.750000	4.916667
$GAS2$	42.000000	8.428572	19.000000
$GAS3$	39.000000	19.666666	13.400000
$PARAM$	1.000000	3.045455	2.269231

6. (from an old exam) We are running a factory and can produce products of three possible types from four types of parts as follows.

	product 1	product 2	product 3	available parts
part 1	3	5	2	286
part 2	4	6	2	396
part 3	5	8	3	440
part 4	4	7	4	396
profit $	21	35	15	

We wish to choose our product mix to obtain maximum profit subject both to the limitations on the inventory of available parts but also subject to the restriction that at most 50% of the number of produced products can be of one type

The LINDO input/output on this page and the next page will be useful for parts a),b),c).

a) What are the marginal values of the four parts?

b) Mr. Edison visits the factory and offers to make a remarkable new part that substitutes for one of any of the four parts and will only charge $2 for each of these new parts. Would you buy some? How many would you buy?

c) The market for product 1 crashes and the profit drops to that of product 3. Should you change your production?

d) Compute the marginal cost for the total parts that make up product 2 and compare with the profit for product 2 (Please let 1.545455 be $\frac{17}{11}$ for this calculation). Why aren't they equal?

e) Explain the meaning of the constraint PROD1<50 in the context of this problem.

The input to LINDO was as follows. The constraints have been labeled to aid readability:

MAX 21 PROD1 + 35 PROD2 + 15 PROD3

SUBJECT TO

PART1) 3 PROD1 + 5 PROD2 + 2 PROD3 < 286

PART2) 4 PROD1 + 6 PROD2 + 2 PROD3 < 396

PART3) 5 PROD1 + 8 PROD2 + 3 PROD3 < 440

PART4) 4 PROD1 + 7 PROD2 + 4 PROD3 < 396

PROD1<50) 0.5 PROD1 - 0.5 PROD2 - 0.5 PROD3 < 0

PROD2<50) - 0.5 PROD1 + 0.5 PROD2 - 0.5 PROD3 < 0

PROD3<50) - 0.5 PROD1 - 0.5 PROD2 + 0.5 PROD3 < 0

END

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

1932.000

VARIABLE	VALUE	REDUCED COST
PROD1	22.000000	0.000000
PROD2	36.000000	0.000000
PROD3	14.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
PART1)	12.000000	0.000000
PART2)	64.000000	0.000000
PART3)	0.000000	3.000000
PART4)	0.000000	1.545455
PROD1<50)	14.000000	0.000000
PROD2<50)	0.000000	0.363636
PROD3<50)	22.000000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
PROD1	21.000000	0.363636	6.000000
PROD2	35.000000	INFINITY	0.500000
PROD3	15.000000	1.333333	2.615385

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
PART1	286.000000	INFINITY	12.000000
PART2	396.000000	INFINITY	64.000000
PART3	440.000000	24.000000	44.000000
PART4	396.000000	44.000000	23.692308
PROD1<50	0.000000	INFINITY	14.000000
PROD2<50	0.000000	22.000000	19.250000
PROD3<50	0.000000	INFINITY	22.000000