In Calculus courses one is often shown reductions of the form

$$\frac{2x+3}{(x-1)(x-2)} = \frac{-5}{x-1} + \frac{7}{x-2}.$$

This is the kind of reduction you need when integrating rational functions of polynomials. The integrals of the simpler expressions on the right are readily (?) seen to be logarithms. The texts that take the time to assert that this reduction is always possible usually make a reference to the students seeing a proof in their linear algebra classes. Most don't but you will.

The general problem, for rational expressions with quadratic denominators, becomes

$$\frac{a_1x+a_0}{(x-r_1)(x-r_2)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2}$$

where we assume $r_1 \neq r_2$ since then no reduction is necessary. We compute

$$\frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} = \frac{(A_1+A_2)x + (-r_2A_1 - r_1A_2)}{(x-r_1)(x-r_2)}.$$

Solving for A_1, A_2 from a_1, a_0 yields the equation

$$\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

Now we check that

$$\det(\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix}) = (r_2 - r_1) \neq 0$$

and so we can always solve for A_1, A_2 from any a_1, a_0.

There are two cases for cubic denominators. First assume there are three distinct roots r_1, r_2, r_3. The following reduction

$$\frac{a_2x^2+a_1x+a_0}{(x-r_1)(x-r_2)(x-r_3)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \frac{A_3}{x-r_3}$$

yields the matrix equation

$$\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -r_2-r_3 & -r_1-r_3 & -r_1-r_2 \\ r_2r_3 & r_1r_3 & r_1r_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

Now we check that

$$\det(\begin{bmatrix} 1 & 1 & 1 \\ -r_2-r_3 & -r_1-r_3 & -r_1-r_2 \\ r_2r_3 & r_1r_3 & r_1r_2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 0 \\ -r_2-r_3 & r_2-r_1 & r_3-r_1 \\ r_2r_3 & (r_1-r_2)r_3 & (r_1-r_3)r_2 \end{bmatrix})$$

$$= (r_2-r_1)(r_3-r_1)(r_3-r_2) \neq 0$$

since $r_1 \neq r_2 \neq r_3$. Hence we can always solve for A_1, A_2, A_3 from any a_2, a_1, a_0.

The second case is that the cubic has a repeated root r_1 and a distinct root r_2:

$$\frac{a_2x^2+a_1x+a_0}{(x-r_1)^2(x-r_2)} = \frac{A_1x+A_2}{(x-r_1)^2} + \frac{A_3}{x-r_2}.$$

This yields the matrix equation

$$\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -r_2 & 1 & -2r_1 \\ 0 & -r_2 & r_1^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

Now we check that

$$\det(\begin{bmatrix} 1 & 0 & 1 \\ -r_2 & 1 & -2r_1 \\ 0 & -r_2 & r_1^2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 0 \\ -r_2 & 1 & r_2 - 2r_1 \\ 0 & -r_2 & r_1^2 \end{bmatrix})$$

$$= r_1^2 + r_2(r_2 - 2r_1) = (r_2 - r_1)^2 \neq 0$$

since $r_1 \neq r_2$. Hence we can always solve for A_1, A_2, A_3 from any a_2, a_1, a_0.

Why are there no other cubic cases?

You can generalize for quartic numerators etc. Try it in the case of a quartic with four distinct roots. *VanderMonde* determinants are of the form

$$\det(\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix}) = -(r_1 - r_2)(r_1 - r_3)(r_2 - r_3).$$

We can see that the matrices

$$\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ -r_2 - r_3 & -r_1 - r_3 & -r_1 - r_2 \\ r_2 r_3 & r_1 r_3 & r_1 r_2 \end{bmatrix}$$

are related by row operations. The second matrix is obtained by the following operations on the first matrix: take $(r_1 r_2 + r_1 r_3 + r_2 r_3)$ times the first row and $(-r_1 - r_2 - r_3)$ times the second row and add to the third row and then take $(-r_1 - r_2 - r_3)$ times the first row and add to the second row.

The general case is

$$\det(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix}) = (-1)^{n(n-1)/2} \prod_{1 \le i < j \le n} (r_i - r_j).$$

$$= \prod_{1 \le i < j \le n} (r_j - r_i)$$