Let U, V be vector spaces with $U \subseteq V$. We consider

$$U^{\perp} = \{\mathbf{v} \in \mathbf{R}^n \ : \ \text{for all } \mathbf{u} \in U, \ <\mathbf{u}, \mathbf{v}>= 0\}$$

Theorem 0.1 *$U^{\perp}$ is a vector space.*

Proof: We show that $U^{\perp}$ is a vector space. Here we must verify that $\mathbf{0} \in U^{\perp}$ since this will not follow from the other two closure rules. We have $\mathbf{0} \in U^{\perp}$ because $<\mathbf{u}, \mathbf{0}>= 0$ always for any choice $\mathbf{u}$. Also if $\mathbf{x}, \mathbf{y} \in U^{\perp}$, then $<\mathbf{x}+\mathbf{y}, \mathbf{u}>=<\mathbf{x}, \mathbf{u}> + <\mathbf{y}, \mathbf{u}>$ and $<c\mathbf{x}, \mathbf{u}>= c<\mathbf{x}, \mathbf{u}>$ by our inner product axioms. Thus if for all $\mathbf{u} \in U$, $<\mathbf{x}, \mathbf{u}>= 0$ and $<\mathbf{y}, \mathbf{u}>= 0$, then we conclude that $<\mathbf{x}+\mathbf{y}, \mathbf{u}>=<\mathbf{x}, \mathbf{u}> + <\mathbf{y}, \mathbf{u}>= 0 + 0 = 0$ and also $<c\mathbf{x}, \mathbf{u}>= c<\mathbf{x}, \mathbf{u}>= c \cdot 0 = 0$. Thus we have $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ in $U^{\perp}$, verifying closure. So $U^{\perp}$ is a vector space. ■

Consider a vector space $U \subseteq \mathbf{R}^n$. Thus we are thinking of $V = \mathbf{R}^n$ with the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for U. Then if we write each $\mathbf{u}_i$ with respect to the standard basis we can form a matrix $A = (a_{ij})$ with the ith row A being $\mathbf{u}_i^T$. Thus row space$(A) = U$ and $\dim(U) = \text{rank}(A)$. Then

$$\text{null space}(A) = \{\mathbf{x} \ : \ A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \ :<\mathbf{x}, \mathbf{u}_i>= 0 \text{ for } i = 1, 2, \ldots, k\}$$

$$= \{\mathbf{x} \ :<\mathbf{x}, \mathbf{u}>= 0 \text{ for all } \mathbf{u} \in U\} = U^{\perp}$$

Here we are assuming $<\mathbf{x}, \mathbf{u}_i>$ is the standard dot product. Thus $\dim(U) + \dim(U^T) = n$ using our result that $\dim(\mathit{nullsp}(A)) + \text{rank}(A) = n$ where n is the number of columns in A.

These ideas will happily generalize to two vector spaces U, V with $U \subseteq V$ with a general inner product. We do not need $V = \mathbf{R}^n$. There are two approaches. The first uses an orthonormal basis for V in order to use the $\mathit{nullsp}(A)$) idea. If we apply Gram Schmidt or otherwise, we can obtain a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ with the orthonormal properties:

$$<\mathbf{v}_i, \mathbf{v}_j>= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad (*)$$

Now proceed much as before, expressing

$$\mathbf{u}_i = \sum_{j=1}^{n} a_{ij}\mathbf{v}_j$$

since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V and $\mathbf{u}_i \in V$. Let A be the associated $k \times n$ matrix. Now consider any vector $\mathbf{w} \in V$ which we can write as $\mathbf{w} = \sum_{j=1}^{n} w_j\mathbf{v}_j$. Let $\mathbf{w}$ denote the vector in the coordinates of the orthonormal basis so $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ Then

$$<\mathbf{u}_i, \mathbf{w}>=<\sum_{j=1}^{n} a_{ij}\mathbf{v}_j, \sum_{\ell=1}^{n} w_\ell \mathbf{v}_\ell>$$

$$= \sum_{j=1}^{n} a_{ij} \left(<\mathbf{v}_j, \sum_{\ell=1}^{n} w_\ell \mathbf{v}_\ell>\right)$$

$$= \sum_{j=1}^{n} a_{ij} \left(\sum_{\ell=1}^{n} w_\ell \left(<\mathbf{v}_j, \mathbf{v}_\ell>\right)\right)$$

$$= \sum_{j=1}^{n} a_{ij} w_j$$

using properties of (*). Now $\sum_{j=1}^{n} a_{ij} w_j$ is the ith entry of $A\mathbf{w}$. Thus we have a way of expressing $U^{\perp}$ as the null space(A) and we have the desired result.

Theorem 0.2 *Let U, V be vector spaces over $\mathbf{R}$ with U a subspace of V and V is finite dimensional. Then* $\dim(U) + \dim(U^{\perp}) = \dim(V)$.

Another approach that doesn't use an orthonormal basis of V (with respect to the given inner product) but just any basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we use the observation that for a given $\mathbf{u}_i$, the function $< \mathbf{u}_i, \mathbf{x} >$ is a linear transformation $V \to \mathbf{R}$ and so has an associated $1 \times n$ matrix R_i. First we verify that the k linear transformations $< \mathbf{u}_i, \mathbf{x} >$ are linearly independent (and so the $k \times n$ matrix formed by these rows has rank $= k$). Assume

$$\sum_{i=1}^{k} c_i < \mathbf{u}_i, \mathbf{x} > \equiv 0$$

where we use the notation $\equiv 0$ to mean the identically 0 function, namely the $\mathbf{0}$ vector in the space of functions. But now

$$\sum_{i=1}^{k} c_i < \mathbf{u}_i, \mathbf{x} > = < \sum_{i=1}^{k} c_i \mathbf{u}_i, \mathbf{x} > \text{ for all } \mathbf{x}$$

but when we evaluate the righthand side at $\mathbf{x} = \sum_{i=1}^{k} c_i \mathbf{u}_i$, we obtain $< \mathbf{x}, \mathbf{x} > = 0$ and so by the axioms of an inner product we have $\mathbf{x} = \mathbf{0}$ i.e. $\sum_{i=1}^{k} c_i \mathbf{u}_i = \mathbf{0}$ which forces $c_1 = c_2 = \cdots = c_k = 0$ since the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent.

We now form a matrix

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix}$$

and null space(A) $= U^{\perp}$.

As an example consider the vector space of polynomials of degree at most 2. Let $U = \text{span} < \mathbf{1}, x >$ and $V = \text{span} < \mathbf{1}, x, x^2 >$. The basis of V will be $\{\mathbf{1}, x, x^2\}$ and the basis for U will be $\{\mathbf{1}, x\}$. We introduce the inner product

$$< f, g > = \int_0^1 f(x) g(x) dx$$

and so we have an inner product space. We have that $< \mathbf{1}, a + bx + cx^2 >$ and $< x, a + bx + cx^2 >$ are linear transformations $T : V \to \mathbf{R}$.

$$< \mathbf{1}, a + bx + cx^2 > = \int_0^1 (a + bx + cx^2) dx = a + \frac{b}{2} + \frac{c}{3} = [1 \;\; 1/2 \;\; 1/3] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$< x, a + bx + cx^2 > = \int_0^1 (ax + bx^2 + cx^3) dx = \frac{a}{2} + \frac{b}{3} + \frac{c}{4} = [1/2 \;\; 1/3 \;\; 1/4] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Thus we take $R_1 = [1\ \ 1/2\ \ 1/3]$ and $R_2 = [1/2\ \ 1/3\ \ 1/4]$ and form

$$A = \left[\begin{array}{ccc} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \end{array}\right].$$

Gaussian elimination yields

$$\text{null space}(A) = \{s \left[\begin{array}{c} 1/6 \\ -1 \\ 1 \end{array}\right] : s \in \mathbf{R}\}$$

We may check that $< 1/6 - x + x^2, \mathbf{1} >= \int_0^1 (1/6 - x + x^2)dx = 1/6 - 1/2 + 1/3 = 0$ and $< 1/6 - x + x^2, x >= \int_0^1 ((1/6)x - x^2 + x^3)dx = 1/12 - 1/3 + 1/4 = 0$ as expected. Thus

Theorem 0.3 *Let U, V be vector spaces with U a subspace of V and V is finite dimensional. Then*

$$U^{\perp\perp} = U.$$

Proof: We note that if $\mathbf{u} \in U$, then $\mathbf{u} \perp \mathbf{v}$ for all $\mathbf{v} \in U^\perp$ and so $\mathbf{u} \in U^{\perp\perp}$, hence $U \subseteq U^{\perp\perp}$. We have that, with $\dim(U) = k$ and $\dim(V) = n$, then $\dim(U^\perp) = n - k$ using the nullity theorem and then $\dim(U^{\perp\perp}) = n - (n-k) = k$. Given that $\dim(U) = \dim(U^{\perp\perp}) = k$, we deduce from $U \subseteq U^{\perp\perp}$ that $U^{\perp\perp} = U$.. ■