Theorem 1 *Let A be an $n \times n$ matrix. The following are equivalent.*
i) A^{-1} exists (we say A is invertible *or* nonsingular*).*
ii) $A\mathbf{x} = \mathbf{0}$ has only one solution, namely $\mathbf{x} = \mathbf{0}$.
iii) A can be transformed to a triangular matrix, with nonzeros on the main diagonal, by elementary row operations.
iv) A can be transformed to I by elementary row operations.

Proof: We can verify by Gaussian elimination that iii)⇒ iv)⇒ i)⇒ ii)⇒ iii), the last implication following because there can be no free variables (the system $A\mathbf{x} = \mathbf{0}$ is always consistent) and so elementary row operations must result in every variable being a pivot variable.

Finding A^{-1}

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

To solve for A^{-1} we can solve

$$A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and it makes sense to solve for all three vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ at the same time:

$$\begin{array}{c} \quad\quad A \quad\quad\quad I \\ \left[\begin{array}{ccc ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{array}{c} E_1A \quad\quad E_1 \\ \left[\begin{array}{ccc ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \end{array}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{array}{c} E_2E_1A \quad\quad E_2E_1 \\ \left[\begin{array}{ccc ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \end{array}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad \begin{array}{c} E_3E_2E_1A \quad\quad E_3E_2E_1 \\ \left[\begin{array}{ccc ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \end{array}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{c} E_5E_4E_3E_2E_1A \quad\quad E_5E_4E_3E_2E_1 \\ \left[\begin{array}{ccc ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \end{array}$$

Thus we have $(E_5E_4E_3E_2E_1)A = I$ and so $A^{-1} = E_5E_4E_3E_2E_1$ and $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$.

By the way, always check your work:

$$A^{-1}A = \begin{bmatrix} 0 & -1 & 1 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$