We use the notation $E(i,j)$ to denote the elementary row operation of interchanging row i and row j. To prove that

$$\det(E(i,j)A) = -\det(A),$$

it suffices to prove that

$$\det(E(1,2)A) = -\det(A).$$

The other cases can be established by induction.

I will offer a proof that involves understanding the row expansion about the first row first applied to A and then applied to each M_{ij}. Let $N_{12,ij}$ denote the matrix obtained from A by deleting rows 1,2 and columns i,j. This will occur in a slightly different order in the recurrence since if we compute M_{1i} then for $j > i$, the jth column of A is the $j-1$st column of M_{1i}.

Case 1. $i > j$

So after an application of row expansion about the first row of A followed by row expansion about the first row of each M_{1i} then the terms involving $\det(N_{12,ij})$ are (in the case $i > j$)

$$(-1)^{1+i}a_{1i}(-1)^{1+j}a_{2j}\det((N_{12,ij}) + (-1)^{1+j}a_{1j}(-1)^{1+i-1}a_{2i}\det((N_{12,ij})$$

$$= ((-1)^{i+j}(a_{1i}a_{2j} - a_{1j}a_{2i})\det(N_{12,ij}).$$

Note the factor $(-1)^{1+i-1}$ since after deleting column j, then column i of A is now column $i-1$ of M_{1j}. Now let $B = (b_{ij}) = E(1,2)A$ so that $b_{1i} = a_{2i}$ and $b_{2i} = a_{1i}$ but also $N_{12,ij}$ is the matrix obtained from B by deleting rows 1,2 and columns i,j. The terms involving $\det(N_{12,ij})$ are

$$(-1)^{1+i}b_{1i}(-1)^{1+j}b_{2j}\det((N_{12,ij}) + (-1)^{1+j}b_{1j}(-1)^{1+i-1}b_{2i}\det((N_{12,ij})$$

$$= (-1)^{1+i}a_{2i}(-1)^{1+j}a_{1j}\det((N_{12,ij}) + (-1)^{1+j}a_{2j}(-1)^{1+i-1}a_{1i}\det((N_{12,ij})$$

$$= ((-1)^{i+j}(a_{2i}a_{1j} - a_{2j}a_{1i})\det(N_{12,ij}) = -((-1)^{i+j}(a_{1i}a_{2j} - a_{1j}a_{2i})\det(N_{12,ij})$$

Thus the coefficients of $\det(N_{12,ij})$ are negatives of each other.

Case 2. $i < j$. Similar result that the coefficients of $\det(N_{12,ij})$ are negatives of each other.

Thus

$$\det(B) = \det(E(1,2)A) = -\det(A)$$

.

Here is another way to organze the proof. For convenience, use the notation M_{ij} to denote the matrix obtained from A by deleting row i and column j. We need to run our recursive definition for det by first expanding about the first row and then for each determinant of M_{ij}, an $(n-1)\times(n-1)$ matrix, expand again about its first row. Thus let $M_{ij,kl}$ to denote the matrix obtained from M_{ij} by deleting row k (of A not M_{ij}) and column l (of A not M_{ij}). This will become complicated as we try to determine the appropriate power of (-1) to use.

Now we expand the two determinants.

$$\det(A) = (-1)^{1+1}a_{11}\det M_{11} + (-1)^{1+2}a_{12}\det M_{12} + \cdots + (-1)^{1+n}a_{1n}\det M_{1n}$$

$$= (-1)^{1+1}a_{11}\Big((-1)^{1+1}a_{22}\det M_{11,22} + (-1)^{1+2}a_{23}\det M_{11,23} + \cdots + (-1)^{1+(n-1)}a_{2n}\det M_{11.2n}\Big)$$

$$+(-1)^{1+2}a_{12}\Big((-1)^{1+1}a_{21}\det M_{12,21} + (-1)^{1+2}a_{23}\det M_{12,23} + \cdots + (-1)^{1+(n-1)}a_{2n}\det M_{12.2n}\Big)$$

$$\vdots$$

$$+(-1)^{1+n}a_{1n}\Big((-1)^{1+1}a_{21}\det M_{1n,21}+(-1)^{1+2}a_{22}\det M_{1n,22}+\cdots+(-1)^{1+(n-1)}a_{2n}\det M_{1n.2(n-1)}\Big)$$

$$=\sum_{1\le i<j\le n}(-1)^{2+i+j-1}(a_{1i}a_{2j}-a_{1j}a_{2i})\det M_{1i,2j}.$$

The last formula is difficult to see. We need the coefficients of $\det M_{1i,2j}$ but note that $M_{1i,2j}=M_{1j,2i}$ are equal as matrices and so $\det(M_{1i,2j})=\det(M_{1j,2i})$. We need only consider products of the form $a_{1i}a_{2j}$ or $a_{1j}a_{2i}$ for a fixed pair i,j with $i<j$. The associated matrix minor in either case is $M_{1i,2j}=M_{1j,2i}$. So the remaining problem is the sign. The term with a_{1i} from the first expansion will have a sign $(-1)^{1+i}$. The subsequent term from expansion about the second row with a_{2j} has a sign $(-1)^{1+(j-1)}$ since column $j-1$ of M_{1i} is column j of A. This yields a sign term $(-1)^{2+i+j-1}$. The term with a_{1j} from the first expansion will have a sign $(-1)^{1+j}$. The subsequent term from expansion about the second row with a_{2i} has a sign $(-1)^{1+i}$ since column i of M_{1j} is column i of A (we are using $i<j$). This yields a sign term of $(-1)^{2+i+j}=-(-1)^{2+i+j-1}$. These two arguments have verified the formula above.

We perform the same task for $E(1,2)A$ where we let $E(1,2)A=(a'_{ij})$ i.e. a'_{ij} is the (i,j) entry of the matrix $E(1,2)A$. Let M'_{ij} denote the matrix obtained from $E(1,2)A$ by deleting its row i and its column j. We define $M'_{ij,kl}$ as the matrix obtained from M'_{ij} by deleting row k (of $E(1,2)A$ not M'_{ij}) and column l (of $E(1,2)A$ not M_{ij}). Given that $E(1,2)A$ is the same as A apart from the first two rows, we see that $M_{1i,2j}=M'_{1i,2j}$. Now we can copy our work from $\det(A)$:

$$\det(E(1,2)A)=(-1)^{1+1}a'_{11}\det M'_{11}+(-1)^{1+2}a'_{12}\det M'_{12}+\cdots+(-1)^{1+n}a'_{2n}\det M'_{1n}$$

$$=\sum_{1\le i<j\le n}(-1)^{2+i+j-1}(a'_{1i}a'_{2j}-a'_{1j}a'_{2i})\det M'_{1i,2j}$$

$$=\sum_{1\le i<j\le n}(-1)^{2+i+j-1}(a_{2i}a_{1j}-a_{2j}a_{1i})\det M_{1i,2j}$$

(using $a'_{1p}=a_{2p}$, $a'_{2q}=a_{1q}$ and $M'_{1p,2q}=M_{1p,2q}$)

$$=-\det(A).$$

I find this argument a bit delicate, but not difficult. We can use summation signs to simplify the writing in the argument above.

$$\det(A)=\sum_{i=1}^{n}(-1)^{1+i}a_{1i}\det M_{1i}$$

$$=\sum_{i=1}^{n}(-1)^{1+i}a_{1i}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n}(-1)^{1+j-\delta}a_{2j}\det M_{1i,2j}\Big)$$

where

$$\delta=\begin{cases}1 & \text{if } j>i\\ 0 & \text{if } j<i\end{cases}.$$

The expression for δ is the same count that was used above e.g when $j>i$, we must subtract one because column $j-1$ of M_{1i} is from column j of A.

Similarly, applying expansion about the first row of $E(1,2)A$ (which is the second row of A), and then expansion about the first rows of the Minors (which will come from the first row of A):

$$\det(E(1,2)A) = \sum_{j=1}^{n}(-1)^{1+j}a_{2j}\det M_{2j}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{2j}\Big(\sum_{\substack{i=1 \\ i\neq j}}^{n}(-1)^{1+i-\delta}a_{1i}\det M_{2j,1i}\Big)$$

where

$$\delta = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i < j \end{cases}.$$

We deduce $\det(A) = -\det(E(1,2)A)$.

There are useful consequences of this result. One basic result that can be obtained is the definition of the signum function for permutations. For a permutation σ of $\{\mathbf{1}, 2, 3, \ldots, m\}$, we say that signum of σ, written $sgn(\sigma)$,is 0 if σ can be obtained as the composition of an even number of interchanges and $sgn(\sigma)$ is 1 if σ can be obtained as the composition of an odd number of interchanges. That this function is well defined follows from the above result (we see that $sgn(\sigma)$ is the determinant of the matrix associated with σ).