MATH 223. Induction.

Induction is the main proof technique for theorems involving or indexed by integers. The positive integers $\mathbf{N} = \mathbb{N} = \{1, 2, \dots\}$ (I'll choose to not include 0) have several properties. The main idea is that if we have s subset X of $\mathbf{N}$ such that $1 \in X$ and for each $s \in X$, we have $s + 1 \in X$, then we may conclude that $X = \mathbf{N}$. This yields induction. Another property is that any subset X of $\mathbf{N}$ will always have a minimum element. This yields the idea of a minimum counterexample which can also be used in proofs. Namely you assume the theorem is false, then consider a minimum counterexample which you prove has various properties that eventually yield some contradiction.

To give an example of induction we need some statement such as $H(n)$ which is the theorem indexed by $n \in \mathbf{N}$. In order to prove $H(n)$ for all $n \in \mathbf{N}$, we only need to prove that $H(1)$ is true as well as show that the truth of $H(s)$ implies the truth of $H(s + 1)$ for each $s \in \mathbf{N}$.

We call proving $H(1)$ the base case and proving $H(s) \implies H(s + 1)$ is the inductive step. An anology with ladders is sometimes helpful. You wish to prove that you can reach any rung of the ladder. Think of the rungs of the ladder as labelled $1, 2, \dots$. First we must get on the ladder (the truth of $H(1)$) and then we need to know that if we are on some rung, say the sth rung then we can step to the $(s + 1)$st rung ($H(s) \implies H(s + 1)$). We then assert we can get to any rung ($H(n)$ is true for any $n \in \mathbf{N}$).

An example for $H(n)$ is important. Here is a somewhat trivial example. No I don't expect you to prove things at this level of detail. This results is quite easy to see. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Theorem 1 *For each $n \in \mathbf{N}$,* $\qquad D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$.

We decide that we can set $H(n)$ to be the statement $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$. We deduce that $H(1)$ is true since

$$D^1 = D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^1 & 0 \\ 0 & 3^1 \end{bmatrix}.$$

(overkill!). Now assume $H(s)$ is true for some $s \in \mathbf{N}$. We wish to show that $H(s + 1)$ is true, namely

$$D^{s+1} = \begin{bmatrix} 2^{s+1} & 0 \\ 0 & 3^{s+1} \end{bmatrix}$$

We have $D^{s+1} = D^s D$ (you will have defined powers of a matrix in this inductive way!) and then we use the truth of $H(s)$ to obtain

$$D^{s+1} = D^s D = \begin{bmatrix} 2^s & 0 \\ 0 & 3^s \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2^s \cdot 2 & 0 \\ 0 & 3^s \cdot 3 \end{bmatrix} = \begin{bmatrix} 2^{s+1} & 0 \\ 0 & 3^{s+1} \end{bmatrix}.$$

Thus we have verified the truth of $H(s + 1)$. Now we can conclude that $H(n)$ is true for all $n \in \mathbf{N}$.

In general applications of induction can be quite a bit more complicated where choosing $H(n)$ is a bit tricky. Also we may have trouble proving $H(s) \implies H(s + 1)$. A variation that is often used, that also works, is that instead of showing $H(s) \implies H(s + 1)$ it is enough to show the the truth of all $H(t)$ for $t \in \{1, 2, \dots, s\}$ will imply $H(s + 1)$. For unclear reasons this is sometimes called *strong induction*. It can be used anytime in place of showing $H(s) \implies H(s + 1)$ if it is helpful.