Some examples. Imagine we have a 3-dimensional vector space $V = \text{span}\{f_1(x), f_2(x), f_3(x)\}$ where $f_1(x) = e^x$, $f_2(x) = e^{2x}$ and $f_3(x) = e^{3x}$. Demonstrating that these three are linearly independent is relatively easy (you could even examine the differing growth rates of the functions to prove linear independence). We can think of $\{f_i(x), f_2(x), f_3(x)\}$ as a basis F for V. We consider the linear transformation $T : V \to V$ defined as

$$T(h(x)) = h(x) + \frac{d}{dx}h(x).$$

We can represent T by a matrix when considering vectors in V written with respect to F.

$$\underset{T \text{ with respect to } F}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

We can consider other coordinate systems for V. Let $g_1(x) = e^x + e^{2x}$, $g_2(x) = e^{2x} + e^{3x}$ and $g_3(x) = e^x + e^{3x}$. We have the following

$$M = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \end{matrix} \underset{F \leftarrow G}{\overset{\begin{matrix} g_1 & g_2 & g_3 \end{matrix}}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}}$$

We can compute

$$M^{-1} = \begin{matrix} g_1 \\ g_2 \\ g_3 \end{matrix} \underset{G \leftarrow F}{\overset{\begin{matrix} f_1 & f_2 & f_3 \end{matrix}}{\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}}}$$

The existence of M^{-1} means that $f_1, f_2, f_3 \in \text{span}\{g_1(x), g_2(x), g_3(x)\}$ and easily we see $\text{span}\{g_1(x), g_2(x), g_3(x)\} \subseteq V$ from which we deduce that $\text{span}\{g_1(x), g_2(x), g_3(x)\} = V$ and so $\{g_1(x), g_2(x), g_3(x)\}$ forms a basis for V. What is T written as a matrix with respect to G?

$$\underset{G \leftarrow F}{\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}} \underset{T \text{ with respect to } F}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \underset{F \leftarrow G}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} = \underset{T \text{ with respect to } G}{\begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 7/2 & 1 \\ -1/2 & 1/2 & 3 \end{bmatrix}}$$

You can check

$$T(g_1 + g_2) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G\right) = \underset{T \text{ with respect to } G}{\begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 7/2 & 1 \\ -1/2 & 1/2 & 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}_G \qquad (1)$$

We note that $g_1(x) + g_2(x) = e^x + 2e^{2x} + e^{3x} = f_1(x) + 2f_2(x) + f_3(x)$ so that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_F$$

We $T(f_1(x) + 2f_2(x) + f_3(x))$ is computed as

$$\underset{T \text{ with respect to } F}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_F = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}_F = 2f_1(x) + 6f_2(x) + 4f_3(x).$$

We compute $2f_1(x)+6f_2(x)+4f_3(x) = 2e^x+6e^{2x}+4e^{3x} = 2(e^x+e^{2x})+4(e^{2x}+e^{3x}) = 2g_1(x)+4g_2(x)$. This is (1) above.