First consider the system of DE's which we motivated in class using water passing through two tanks while flushing out salt contamination.

$$\begin{array}{rcrcr} y_1'(t) & = & -\frac{1}{10}y_1(t) & + & \frac{1}{40}y_2(t) \\ y_2'(t) & = & \frac{1}{10}y_1(t) & - & \frac{1}{10}y_2(t) \end{array}, \qquad y_1(0)=60, y_2(0)=0$$

$$\frac{d}{dt}\mathbf{y} = A\mathbf{y} \text{ where } A = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}$$

We may compute

$$\begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -3/20 & 0 \\ 0 & -1/20 \end{bmatrix} \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix} = MDM^{-1}$$

I offer several solutions, the first using change of variable (change of basis). The second considers the matrix exponential ($\mathbf{y} = e^{At}\mathbf{y}(0)$) and the third solution considers write the final solution as a linear combination of solutions to the DE to satisfy the initial conditions.

Our first idea was to rewrite $\frac{d}{dt}\mathbf{y} = A\mathbf{y}$ as $\frac{d}{dt}\mathbf{y} = MDM^{-1}\mathbf{y}$ and then $M^{-1}\frac{d}{dt}\mathbf{y} = DM^{-1}\mathbf{y}$. Then using the linearity of differentiation, we have $\frac{d}{d}M^{-1}\mathbf{y} = DM^{-1}\mathbf{y}$. We set

$$\mathbf{z} = M^{-1}\mathbf{y} \text{ and so } \mathbf{y} = M\mathbf{z}$$

and obtain the 'easy' system of differential equations

$$\frac{d}{dt}\mathbf{z} = D\mathbf{z}$$

namely

$$\frac{d}{dt}z_1(t) = (-3/20)z_1(t),$$
$$\frac{d}{dt}z_2(t) = (-1/20)z_2(t),$$

which we solve as
$z_1(t) = z_1(0)e^{(-3/20)t}$, $\quad z_2(t) = z_2(0)e^{(-1/20)t}$.

We may compute $z_1(0), z_2(0)$ using $y_1(0) = 60$ and $y_2(0) = 0$ and $\mathbf{z} = M^{-1}\mathbf{y}$ to obtain $z_1(0) = 30$ and $z_2(0) = 30$. Now we use $\mathbf{y} = M\mathbf{z}$ and obtain

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 30e^{(-3/20)t} \\ 30e^{(-1/20)t} \end{bmatrix} = \begin{bmatrix} 30e^{(-3/20)t} + 30e^{(-1/20)t} \\ -60e^{(-3/20)t} + 60e^{(-1/20)t} \end{bmatrix}$$

TThe term involving $e^{(-1/20)t}$ dominates. This solution technique of changing variables (changing basis) to make the system of differential equations easy to solve (diagonalization) follows our usual pattern.

A second solution involves tackling the problem directly in what first appears a bit unlikely:

$$\frac{d}{dt}\mathbf{y} = A\mathbf{y}, \qquad \mathbf{y} = e^{At}\mathbf{y}(0)$$

Recall that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$\frac{d}{dt}e^{At} = 0 + A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$$

where the derivative has been done entrywise. We note that e^{At} at $t = 0$ is in fact $e^0 = I$ (we are exponentiating the 2×2 zero matrix) and hence $e^{At}\mathbf{y}(0)$ at $t = 0$ is indeed $\mathbf{y}(0)$ as desired.

We have techniques for doing this namely:

$$\mathbf{y} = e^{At}\mathbf{y}(0) = Me^{Dt}M^{-1}\mathbf{y}(0)$$

This technique can be used for non diagonalizable matrices A if we can find a similar matrix B (i.e. $A = MBM^{-1}$) for which e^{Bt} is easy to compute.

A third solution technique is commonly used when solving DE's, we write our solution in vector form in terms of the eigenvectors

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 30e^{(-3/20)t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 30e^{(-1/20)t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This suggests another solution strategy. We seek solutions of the form

$e^{\lambda t}\mathbf{v}$ with $\frac{d}{dt}e^{\lambda t}\mathbf{v} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$ by solving $A\mathbf{v} = \lambda\mathbf{v}$ and hence solving for eigenvalues and eigenvectors of A. Then (it needs to be proven!) we write an arbitrary solution to the system of DE's as

$$\mathbf{y} = ae^{(-3/20)t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + be^{(-1/20)t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and solving for a, b given the values $\mathbf{y}(0)$.

Consider the system

$$\begin{array}{rcrcl} y_1'(t) & = & & + & y_2(t) \\ y_2'(t) & = & -4y_1(t) & + & 4y_2(t) \end{array}, \qquad y_1(0) = 60, y_2(0) = 0$$

We begin by noticing the following:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

We can solve the system of DE's by the matrix exponential idea: $\mathbf{y}(t) = e^{At}\mathbf{y}(0) = Me^{St}M^{-1}\mathbf{y}(0)$. For this we need to have a way to compute S^n and then e^{tS} which is an assignment question.