**MATH 184: Differential Calculus with applications to Commerce and the Social Sciences**

Some comments on limits and $2^x$

I gave an example in the lecture of September 13 concerning limits and the function $2^x$. For most of you the function was given to you and you accepted it. If nothing else your caculator returned a value for any $x$ you entered. My comments about $2^x$ were to give one way to define $2^x$ for any $x$ as well as give an example how limits can be applied in useful and fundamental ways. The details are not testable.

First you are familiar with $2^n$ for $n$ a positive integer. e.g. $2^4 = 2 \times 2 \times 2 \times 2$. We readily accept that we can extend this to all positive integers $n$. Then we extend to all integers using the rule that $2^{-x} = \frac{1}{2^x}$. This rule can be viewed as the consequence of requiring $2^{x+y} = 2^x 2^y$. Now we need a way to interpret $2^{1/m}$. We have that $2^{1/2} = \sqrt{2}$ since $2^{1/2}2^{1/2} = 2^{1/2+1/2} = 2^1 = 2$. Similarly $2^{1/3} = \sqrt[3]{2}$. Thus for a positive integer $m$, we have that $2^{1/m}$ is the $m$th root of 2. This gives us the way to compute $2^{n/m}$ where $m, n$ are integers and $m$ is positive. We have $2^{n/m} = \left(2^{1/m}\right)^n$.

Now any real number $x$ can be closely approximated by rational numbers. This yields the following definition:

$$2^x = \lim_{\frac{n}{m} \to x} 2^{\frac{n}{m}}.$$

Now of course we can find various rational approximations to $x$, perhaps by using a decimal expansion for $x$. Say $x = 3.1415926\ldots$, Then a first approximation is 3, then we could use 3.1, then we could use 3.14, then we could use 3.141, then we could use 3.1415 etc. Each finite decimal expression is a rational (e.g. $3.1415 = \frac{31415}{10000}$). It remains a reasonable question whether the limit exists. We could at least note that for $n/m$ close to $n'/m'$ then we have $2^{n/m}$ is close to $2^{n'/m'}$. This will be enough but any further details should be left to another course!

What we have done is shown how the whole function $2^x$ comes from limits (one for each irrational $x$ since we already know how to handle rational $x$). Later in the course we could give a different approach, again using limits, namely Taylor approximations. Now how do we deal with $e^x$ where in fact $e$ is also given by a limit?