MATH 184 Runners problem

Intermediate Value Theorem

I was told this problem while running with Claude Belisle, a professor at Université Laval. It is published in a paper: Le Problème du coureur et son interprétation probabiliste, Ann. Sci. Math. Québec, **19**(1995), 1-8.

We consider the situation of a runner who finished a 12km race in 48 minutes. This is on average 4 minutes per km. The runner is interested in running in an upcoming 10km race and wonders if he can do it in 40 minutes.

Question 1. Did he complete a 10 km race in exactly 40 minutes in a 10 km segment of his 12 km race?

being Mathematically inclined another question gets asked

Question 2. Did he complete a 6 km race in exactly 24 minutes in a 6 km segment of his 12 km race?

The answer to Question 2 is Yes and the answer to Question 1 is no (or in particular, not necessarily).

For Question 2 we define a function $f(x)$ that considers the various 6 km segments that could be considered. We define $f(x)$ = time required to run from $x$km to $x+6$km segment of 12 km race. Thus f has domain $[0,6]$. We observe that f is continuous and so can apply the Intermediate Value Theorem.

We consider $f(0)$ and $f(6)$. We note that $f(0)+f(6)=48$ since the two 6 km segments covers the entire 12 km race. Now if $f(0)=24$ we are done and have Yes answer. If $f(0)>24$, then $f(6)<24$ using $f(0)+f(6)=48$. But now we may appeal to the Intermediate Value Theorem to deduce there is some $c\in[0,6]$ with $f(c)=24$. Thus we have run the 6 km (from c km to $c+6$ km) in 24 minutes answering Yes to Question 2

For Question 1 the following (reasonable) running pattern demonstrates a case where the asnwer in No. Run the first km in 3 minutes, the next 10 kms in 42 minutes at a constant pace and the last km in 3 minutes. As for question 1, we may define a continuous function $g(x)$ = time required to run from $x$km to $x+10$km segment of 12 km race. Thus g has domain $[0,2]$. But for our given running pattern, $g(x)>24$ for all x in the domain. e.g. $g(0)=3+9\times 4.2=40.8$ minutes. This yields No to Question 1 (at least we cannot conclude Yes).