Solutions to Problems on the Newton-Raphson Method

These solutions are not as brief as they should be: it takes work to be brief. There will, almost inevitably, be some numerical errors. Please inform me of them at `adler@math.ubc.ca`. We will be excessively casual in our notation. For example, $x_3 = 3.141592654$ will mean that the calculator gave this result. It does not imply that x_3 is exactly equal to 3.141592654.

We should always treat at least the final digit of a calculator answer with some skepticism. Indeed different calculators can give (mildly) different answers. In applied work, we need to pay heed to the fact that the standard tools, such as calculators and computer programs, work only to limited precision. In a complex calculation, minor inaccuracies may result in a significant error.

1. Use the Newton-Raphson method, with 3 as starting point, to find a fraction that is within 10^{-8} of $\sqrt{10}$. Show (without using the square root button) that your answer is indeed within 10^{-8} of the truth.

   Solution: The number $\sqrt{10}$ is the unique positive solution of the equation $f(x) = 0$ where $f(x) = x^2 - 10$. We use the Newton Method to approximate a solution of this equation.

   Let x_0 be our initial estimate of the root, and let x_n be the n-th improved estimate. Note that $f'(x) = 2x$. The Newton Method recurrence is therefore

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 10}{2x_n}.$$

   To make the expression on the right more beautiful, and calculations easier, it is useful to manipulate it a bit. We get

$$x_{n+1} = x_n - \frac{x_n}{2} + \frac{10}{2x_n} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right).$$

Compute, starting with $x_0 = 3$. Then $x_1 = (1/2)(x_0 + 10/x_0) = (1/2)(3 + 10/3) = 19/6$. And $x_2 = (1/2)(19/6 + 60/19) = 721/228$. We could go on calculating with fractions—and there is interesting mathematics involved—but from here on we switch to the calculator.

If we allow the = sign to be used sloppily, we get $x_1 = 3.166666667$. Then $x_2 = (1/2)(x_1 + 10/x_1) = 3.162280702$, and $x_3 = 3.16227766$, and $x_4 = 3.16227766$.

The calculator says that $x_3 = x_4$ to 8 decimal places. We can therefore dare hope that 3.16227766 is close enough. One way of checking is to let $a = 3.16227765$ and $b = 3.16227767$. A quick calculation shows—if the squaring button can be trusted, and it is one of the ones that can be—that $f(a) < 0$ while $f(b) > 0$.

Thus the function $f(x)$ changes sign as x goes from a to b. It follows by the Intermediate Value Theorem that $f(x) = 0$ has a solution (namely $\sqrt{10}$) between a and b. Since $\sqrt{10}$ lies in the interval (a, b), and the distance from 3.16227766 to either a or b is 10^{-8}, it follows that the distance from 3.16227766 to $\sqrt{10}$ is less than 10^{-8}.

2. Let $f(x) = x^2 - a$. Show that the Newton Method leads to the recurrence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Heron of Alexandria (60 CE?) used a pre-algebra version of the above recurrence. It is still at the heart of computer algorithms for finding square roots.

Solution: We have $f(x) = 2x$. The Newton Method therefore leads to the recurrence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}.$$

Bring the expression on the right hand side to the common denominator $2x_n$. We get

$$x_{n+1} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

3. Newton's equation $y^3 - 2y - 5 = 0$ has a root near $y = 2$. Starting with $y_0 = 2$, compute y_1, y_2, and y_3, the next three Newton-Raphson estimates for the root.

Solution: Let $f(y) = y^3 - 2y - 5$. Then $f'(y) = 3y^2 - 2$, and the Newton Method produces the recurrence

$$y_{n+1} = y_n - \frac{y_n^3 - 2y_n - 5}{3y_n^2 - 2} = \frac{2y_n^3 + 5}{3y_n^2 - 2}$$

(there was no good case for simplification here). Start with the estimate $y_0 = 2$. Then $y_1 = 21/10 = 2.1$. It follows that (to calculator accuracy) $y_2 = 2.094568121$ and $y_3 = 2.094551482$. These are almost the numbers that Newton obtained (see the notes). But Newton in effect used a rounded version of y_2, namely 2.0946.

4. Find all solutions of $e^{2x} = x + 6$, correct to 4 decimal places; use the Newton Method.

   Solution: Let $f(x) = e^{2x} - x - 6$. We want to find where $f(x) = 0$. Note that $f'(x) = 2e^{2x} - 1$, so the Newton Method iteration is

$$x_{n+1} = x_n - \frac{e^{2x_n} - x_n - 6}{2e^{2x_n} - 1} = \frac{(2x_n - 1)e^{2x_n} + 6}{2e^{2x_n} - 1}.$$

   We need to choose an initial estimate x_0. This can be done in various ways. We can (if we are rich) use a graphing calculator or a graphing program to graph $y = f(x)$ and eyeball where the graph crosses the x-axis. Or else, if (like the writer) we are poor, we can play around with a cheap calculator, a slide rule, an abacus, or scrap paper and a dull pencil.

   It is easy to verify that $f(1)$ is about 0.389, and that $f(0.95)$ is about -0.2641, so by the Intermediate Value Theorem there is a root between 0.95 and 1. And since $f(0.95)$ is closer to 0 than is $f(1)$, maybe the root is closer to 0.95 than to 1. Let's make the initial estimate $x_0 = 0.97$.

   The calculator then gives $x_1 = 0.970870836$, and then $x_2 = 0.97087002$. Since these two agree to 5 decimal places, we can perhaps conclude with some (but not complete) assurance that the root, to 4 decimal places, is 0.9709. If we want greater assurance, we can compute $f(0.97085)$ and $f(0.97095)$ and hope for a sign change, which shows that there is a root between 0.97085 and 0.97095. There is indeed such a sign change: $f(0.97085)$ is about -2.6×10^{-4} while $f(0.97095)$ is about 10^{-3}.

   But the problem asked for *all* the solutions. Are there any others?

Draw the graphs of $y = e^{2x}$ and $y = x + 6$. The solutions of our equation are the x-coordinates of all places where the two curves meet. Even a rough picture makes it clear that the curves meet at some negative x. Since e^{2x} decays quite rapidly as x decreases through negative values, it seems reasonable that there will be a single negative root, barely larger than -6. Certainly it cannot be smaller, since to the left of -6, $x+6$ is negative but e^{2x} is not. And it seems plausible that the positive root we found is the only one.

We first estimate the negative root. It is reasonable to start with $x_0 = -6$. Then $x_1 = -5.999993856$. We can guess that the root is indeed -6 to 4 decimal places. For certainty, we should check that $f(x)$ has different signs at -6 and -5.9999. It does.

Let's check that there are no more roots. Note that $f'(x) = 2e^{2x} - 1$. Thus $f'(x) > 1$ when $x > 0$, and in particular f is increasing from 0 on, indeed it starts increasing at $x = -(1/2)\ln(2)$. Note also that $f(0) < 0$, and that, for example, $f(1) > 0$. So there is at least one root r between 0 and 1. But there can only be one root there. For $f(x)$ is increasing in the first quadrant, so can cross the x-axis only once.

A similar argument shows that there is a single negative root. For since $f(x)$ is negative in the interval $(-\infty, (1/2)\ln 2)$, the function f is decreasing in this interval, so can cross the x-axis at most once in this interval. We saw already that it crosses the x-axis near $x = -6$.

Note. There are many other ways of solving the problem. For example our equation is equivalent to $2x = \ln(x+6)$, and we could apply the Newton Method to $2x - \ln(x+6)$. Or we can use basically the same approach as above, but let $y = 2x$. We end up solving $e^y = y/2 + 6$. If we are doing the calculations by hand, this saves some arithmetic.

5. Find all solutions of $5x + \ln x = 10000$, correct to 4 decimal places; use the Newton Method.

   Solution: Let $f(x) = 5x + \ln x - 10000$. We need to approximate the root(s) of the equation $f(x) = 0$. The function f is only defined for positive x. Note that the function is steadily increasing, since $f'(x) = 5 + 1/x > 0$ for all positive x. It follows that the function can be 0 for at most one value of x. It is easy to verify that $f(1) < 0$ and $f(2000) > 0$, and therefore the equation has a root in the interval $(1, 2000)$.

The Newton Method iteration is easy to set up. We get

$$x_{n+1} = x_n - \frac{5x_n + \ln x_n - 10000}{5 + 1/x_n}.$$

We could simplify the right hand side somewhat. This is probably not worthwhile.

Now we need to choose x_0. The idea is that even when x is large, $\ln x$ is by comparison quite small. So as a first approximation we can forget about the $\ln x$ term, and decide that $f(x)$ is approximately $5x - 10000$. Thus the root of our original equation must be near $x = 2000$.

Shall we choose $x_0 = 2000$? It is sensible to do so. But we can do better. Note that $\ln(2000)$ is about 7.6. So we can take $5x_0 \approx 10000 - 7.6$. Let $x_0 = 1998.48$.

A quick computation gives $x_1 = 1998.479972$. This agrees with x_0 to 4 decimal places, so the answer, correct to 4 decimal places, should be 1998.4800. If we feel like it, we can show by the usual "sign change" procedure that this answer is indeed correct to 4 places.

Note. If we start with $x_0 = 2000$, it turns out that $x_1 = 1998.479972$, so perhaps the extra thinking that went into starting with 1998.48 was unnecessary. But it illustrates the fact that in some cases we can get an extremely accurate estimate of a root without bringing out heavy machinery.

6. A calculator is defective: it can only add, subtract, and multiply. Use the equation $1/x = 1.37$, the Newton Method, and the defective calculator to find $1/1.37$ correct to 8 decimal places.

   Solution: For convenience we write a instead of 1.37. Then $1/a$ is the root of the equation

   $$f(x) = 0 \quad \text{where} \quad f(x) = a - \frac{1}{x}.$$

   We have $f'(x) = 1/x^2$, and therefore the Newton Method yields the iteration

   $$x_{n+1} = x_n - \frac{a - 1/x_n}{1/x_n^2} = x_n - x_n^2(a - 1/x_n) = x_n(2 - ax_n).$$

   Note that the expression $x_n(2 - ax_n)$ can be evaluated on our defective calculator, since it only involves multiplication and subtraction.

Pick x_0 reasonably close to $1/1.37$. The choice $x_0 = 1$ would work out fine, but I will start off a little closer, maybe by noting that 1.37 is about 4/3 so its reciprocal is about 3/4. Choose $x_0 = 0.75$. We will report answers as they come out of the calculator.

We get $x_1 = x_0(2 - 1.37x_0) = 0.729375$. Thus $x_2 = 0.729926589$, and $x_3 = 0.729927007$. And it turns out that $x_4 = x_3$ to the 9 decimal places that my calculator shows. So we can be reasonably confident that $1/1.37$ is equal to 0.72992701 to 8 decimal places.

I went out and spent almost $9 on a calculator with a "$1/x$" button. It tells me that $1/1.37$ is indeed equal to x_3 to 9 decimal places. But it was not necessary to spend all that money. To check that 0.72992701 is correct to 8 decimal places, it is enough to check by multiplication that $(1.37)(0.729927005) < 1$ and $(1.37)(0.729927015) > 1$.

Note. In the early days of computing, the technique for finding $1/a$ described above was of great practical importance. Computers had addition, subtraction, and multiplication "hard-wired." But division was not hard-wired, and had to be done by software. Note that $x/y = x(1/y)$, so if multiplication is hard-wired, we can do division if we can find reciprocals. And Newton's Method was used to do that.

7. (a) A devotee of Newton-Raphson used the method to solve the equation $x^{100} = 0$, using the initial estimate $x_0 = 0.1$. Calculate the next five Newton Method estimates.

   (b) The devotee then tried to use the method to solve $3x^{1/3} = 0$, using $x_0 = 0.1$. Calculate the next ten estimates.

Solution (a) Let $f(x) = x^{100}$. Then $f'(x) = 100x^{99}$ and the Newton Method iteration is

$$x_{n+1} = x_n - \frac{x_n^{100}}{100x_n^{99}} = \frac{99}{100}x_n.$$

So, to calculator accuracy, $x_1 = 0.099$, $x_2 = .09801$, $x_3 = 0.0970299$, $x_4 = 0.096059601$, and $x_5 = 0.095099004$.

Note the slow progress rate. The root is 0, of course, but in 5 steps we have barely inched closer to the truth.

(b) Let $f(x) = 3x^{1/3}$. Then $f'(x) = x^{-2/3}$, and the Newton Method iteration becomes

$$x_{n+1} = x_n - \frac{3x^{1/3}}{x^{-2/3}} = x_n - 3x_n = -2x_n.$$

Now everything is easy. The next 10 estimates are -0.2, 0.4, -0.8, 1.6, -3.2, 6.4, -12.8, 2.56, -5.12, and 10.24. It is obvious that things are going bad. In fact, if we start with *any* non-zero estimate, the Newton Method estimates oscillate more and more wildly.

Note. The above two examples—with very slow convergence in (a) and total failure in (b)—are not at all typical. Ordinarily the Newton Method is marvellously efficient, at least if the initial estimate is close enough to the truth.

Note that in part (a), successive estimates were quite close to each other, but not really close to the truth. So we need to be a little cautious about the usual rule of thumb that we can stop when two successive estimates agree to the number of decimals we are interested in. But still, in most cases, the rule of thumb is a good one.

8. Suppose that

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function f is continuous everywhere, in fact differentiable arbitrarily often everywhere, and 0 is the only solution of $f(x) = 0$. Show that if $x_0 = 0.0001$, it takes more than one hundred million iterations of the Newton Method to get below 0.00005.

Solution: The differentiation (for $x \neq 0$) is straightforward. (Showing that $f'(0) = 0$ is more delicate, but we don't need that here.) By the Chain Rule,

$$f'(x) = \frac{2e^{-1/x^2}}{x^3}.$$

Write down the standard Newton Method iteration. The e^{-1/x_n^2} terms cancel, and we get

$$x_{n+1} = x_n - \frac{x_n^3}{2} \quad \text{or equivalently} \quad x_n - x_{n+1} = \frac{x_n^3}{2}.$$

Now the analysis is somewhat delicate. It hinges on the fact that if x_n is close to 0, then x_{n+1} is very near to x_n, meaning that each iteration gains us very little additional accuracy.

Start with $x_0 = 0.0001$. It is fairly easy to see that $x_n > 0$ for all n. For $x_1 = x_0(1 - x_0^2/2)$, and in particular $0 < x_1 < x_0$. The same idea shows that $0 < x_2 < x_1$, but then $0 < x_3 < x_2$, and so on forever.

Thus if we start with $x_0 = 0.0001$, the difference $x_n - x_{n+1}$ will always be positive and equal to $x_n^3/2$, and in particular less than or equal to $(0.0001)^3/2$. So with each iteration there is a shrinkage of at most 5×10^{-13}. But to get from 0.0001 to 0.00005 we must shrink by more than 5×10^{-5}. Thus we will need more than $(5 \times 10^{-5})/(5 \times 10^{-13})$, that is, 10^8 iterations. (More, because as we get closer to 0.00005, the shrinkage per iteration is less than what we estimated.)

9. Use the Newton Method to find the smallest and the second smallest positive roots of the equation $\tan x = 4x$, correct to 4 decimal places.

Solution: Draw the curves $y = \tan x$ and $y = 4x$. The roots of our equation are the x-coordinates of the places where these two curves meet.

A glance at the picture shows that (for $x \geq 0$) the curves meet at $x = 0$, then at a point with x just shy of $\pi/2$, and then again at a point with x just shy of $3\pi/2$ (the pattern continues).

We first find the root that is near $\pi/2$. Let $f(x) = \tan x - 4x$. The $f'(x) = \sec^2 x - 4$, and the Newton Method recurrence is

$$x_{n+1} = x_n - \frac{\tan x_n - 4x_n}{\sec^2 x_n - 4}.$$

Some simplification is possible. For example, we can use the identity $\sec^2 x = 1 + \tan^2 x$ to rewrite the recurrence as

$$x_{n+1} = x_n - \frac{\tan x_n - 4x_n}{\tan^2 x_n - 3}.$$

This trick cuts down on the computational work. This was a particularly important consideration in the old days when computations were done by hand, with the aid of tables and slide rules.

For the first root, a bit of fooling around suggests taking $x_0 = 1.4$. Then $x_1 = 1.393536477$, $x_2 = 1.393249609$, and $x_3 = 1.393249075$. This suggests that to 4 decimal places the root is 1.3932. We can verify this by the sign change criterion in the usual way.

For the second root, after some work we can for example arrive at the initial estimate $x_0 = 4.66$. The computation is *quite* sensitive to the right choice of initial value. And then we get $x_1 = 4.658806388$ and $x_2 = 4.658778278$. To 4 decimal places the root is 4.6588. We can verify that we are close enough by the sign change criterion.

Note. We may be nervous about using a casual sketch to locate the first two positive roots. If we are, we can analyze the behaviour of $f(x)$ by looking at its derivative $f'(x)$. Recall that $f'(x) = \sec^2 x - 4$. The sec function increases steadily in the interval $(0, \pi/2)$. It follows that $f'(x)$ is negative in this interval up to the point where $f'(x) = 0$, which happens when $\sec x = 2$, that is, when $\cos x = 1/2$, at $\pi/3$. So $f(x)$ decreases from $x = 0$ to $x = \pi/3$, then increases. Since $f(0) = 0$, we conclude that f is negative in the interval $(0, \pi/3)$, then increases. It becomes very large positive near $x = \pi/2$. So $f(x) = 0$ in exactly one place in the interval $(0, \pi/2)$. In a similar way, we can show that $f(x) = 0$ at exactly one place in the interval $(\pi/2, 3\pi/2)$.

Note. We attacked the problem in the 'natural' way, making the most obvious choice for $f(x)$. It turns out that, particularly for the second positive root, and even more so say for the fifth positive root, we have to be very careful in our choice of x_0. The problem is that near the roots the tan function is growing at a violent rate. A quite small change in x can have a dramatic effect on $\tan x$.

We can rewrite the equation $\tan x = 4x$ in ways that avoid most of the problems. For instance, rewrite it as $g(x) = 0$ where $g(x) = \sin x - 4x \cos x$. The Newton-Raphson recurrence becomes

$$x_{n+1} = x_n - \frac{\sin x_n - 4x_n \cos x_n}{4x_n \sin x_n - 3 \cos x_n}.$$

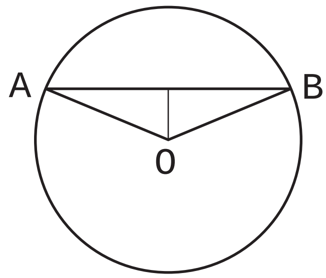
Calculations with this recurrence are quite a bit more numerically stable than calculations with the 'natural' recurrence that we used earlier.

10. The circle below has radius 1, and the *longer* circular arc joining A and B is twice as long as the chord AB. Find the length of the chord AB, correct to 18 decimal places.

Solution: This is somewhat of a trick question. Sorry! It seemed like a good idea at the time.

Draw a perpendicular from O to AB, meeting AB at M. Let $\theta = \angle AOM$. Standard trigonometry shows that the length of AB is $2\sin\theta$. The shorter arc joining A and B has length 2θ, so the longer arc has length $2\pi - 2\theta$. The longer arc is twice the chord, and therefore

$$2\pi - 2\theta = 4\sin\theta.$$



We can use the Newton Method to solve this equation as it stands. Let $f(\theta) = 2\sin\theta + \theta - \pi$. Then $f'(\theta) = 2\cos\theta + 1$, and the Newton Method recurrence is

$$\theta_{n+1} = \theta_n - \frac{2\sin\theta_n + \theta_n - \pi}{2\cos\theta_n + 1}.$$

An ordinary calculator will only handle this to 8 or 9 places. The scientific calculator that comes bundled with Microsoft Windows can handle about 30.

But we can find the answer without doing any work by looking back at a calculation done in the notes. We are solving $\pi - \theta = 2\sin\theta$. Note that by the symmetry of the sine function, we have $\sin\theta = \sin(\pi - \theta)$. Let $x = \pi - \theta$. Then our equation is equivalent to $x = 2\sin x$.

It so happens that in the notes this equation is solved to high accuracy. The positive root of this equation, to 19 places, is given there as $x = 1.8954942670339809471$. But the length of the chord is $2\sin\theta$, that is, $2\sin x$, and that is equal to x.

11. Find, correct to 5 decimal places, the x-coordinate of the point on the curve $y = \ln x$ which is closest to the origin. Use the Newton Method.

    Solution: Let $(x, \tan x)$ be a general point on the curve, and let $S(x)$ be the square of the distance from $(x, \tan x)$ to the origin. Then

    $$S(x) = x^2 + \ln^2 x.$$

    We want to minimize the distance. This is equivalent to minimizing the square of the distance. Now the minimization process takes the usual route. Note that $S(x)$ is only defined when $x > 0$. We have

    $$S'(x) = 2x + 2\frac{\ln x}{x} = \frac{2}{x}(x^2 + \ln x).$$

Since $\ln x$ is increasing, and x^2 is increasing for $x > 0$, it follows that $S'(x)$ is always increasing. It is clear that $S'(x) < 0$ for a while, for example at $x = 1/2$, and that $S'(1) > 0$. It follows that $S'(x)$ must be 0 at some place r between $1/2$ and 1, and that $S'(x) < 0$ if $x < r$ and $S'(x) > 0$ for $x > r$. We conclude that $S(x)$ is decreasing up to $x = r$ and then increasing. Thus the minimum value of $S(x)$, and hence of the distance, is reached at $x = r$.

Our problem thus comes down to solving the equation $S'(x) = 0$. We can use the Newton Method directly on $S'(x)$, but calculations are more pleasant if we observe that $S'(x) = 0$ is equivalent to $x^2 + \ln x = 0$. Let $f(x) = x^2 + \ln x$. Then $f'(x) = 2x + 1/x$ and we get the recurrence

$$x_{n+1} = x_n - \frac{x_n^2 + \ln x_n}{2x_n + 1/x_n}.$$

We need to find a suitable starting point x_0. Experimentation with a calculator suggests that we take $x_0 = 0.65$. Then $x_2 = 0.6529181$, and $x_2 = 0.65291864$. Since x_1 agrees with x_2 to 5 decimal places, we can perhaps decide that, to 5 places, the minimum distance occurs at $x = 0.65292$. If we have doubt, we can try to see whether $f(x)$ has different signs at 0.652915 and 0.652925. It does.

12. It costs a firm $C(q)$ dollars to produce q grams per day of a certain chemical, where

$$C(q) = 1000 + 2q + 3q^{2/3}$$

The firm can sell any amount of the chemical at \$4 a gram. Find the break-even point of the firm, that is, how much it should produce per day in order to have neither a profit nor a loss. Use the Newton Method and give the answer to the nearest gram.

Solution: If we sell q grams then the revenue is $4q$. The break-even point is when revenue is equal to cost, that is, when

$$4q = 1000 + 2q + 3q^{2/3}.$$

Let $f(q) = 2q - 3q^{2/3} - 1000$. We need to solve the equation $f(q) = 0$. It is worth asking first whether there *is* a solution, and whether possibly there might be more than one.

Note that $f(q) < 0$ for "small" values of q, indeed up to 500 and beyond. Also, $f(1000) > 0$. Since f is continuous, it is equal to 0 somewhere between 500 and 1000. We have $f'(q) = 2 - q^{-1/3}$. From this we conclude easily that f is increasing from $q = 1/8$ on. It follows that $f(q) = 0$ at exactly one place.

The Newton Method yields the recurrence

$$q_{n+1} = q_n - \frac{2q_n - 3q_n^{2/3} - 1000}{2 - 2q_n^{-1/3}} = \frac{q_n + 1000q_n^{1/3}}{2q_n^{1/3} - 2}.$$

The simplification is not necessary, but it makes subsequent calculations a bit easier.

Where shall we start? A small amount of experimentation suggests taking q_0 to be say 600. We then get $q_1 = 607.6089386$, and $q_2 = 607.6067886$, perhaps enough to conclude that to the nearest integer the answer is 608. It is, for $f(607.5)$ and $f(608.5)$ have different signs.

Note. The problem is slightly easier to handle if we make the substitution $q = x^3$. Then we end up trying to solve the equation $g(x) = 0$ where $g(x) = 2x^3 - 3x^2 - 1000$. That way we avoid the unpleasantness of dealing with fractional exponents.

13. A loan of A dollars is repaid by making n equal monthly payments of M dollars, starting a month after the loan is made. It can be shown that if the monthly interest rate is r, then

$$Ar = M\left(1 - \frac{1}{(1+r)^n}\right).$$

A car loan of \$10000 was repaid in 60 monthly payments of \$250. Use the Newton Method to find the monthly interest rate correct to 4 significant figures.

Solution: Even quite commonplace money calculations involve equations that cannot be solved by 'exact' formulas. Let r be the interest rate. Then

$$10000r = 250\left(1 - \frac{1}{(1+r)^{60}}\right).$$

If we are going to work by hand, it is maybe worthwhile to simplify a bit to $f(r) = 0$ where

$$f(r) = 40r + \frac{1}{(1+r)^{60}} - 1$$

and therefore

$$f'(r) = 40 - \frac{60}{(1+r)^{61}}.$$

The Newton Method iteration is now easy to write down. In raw form it is

$$r_{n+1} = r_n - \frac{40r_n + 1/(1+r_n)^{60} - 1}{40 - 60/(1+r_n)^{61}}.$$

Compute. Particularly if we do the work by hand, it is helpful to make a good choice of r_0. If the interest rate were 2.5% a month, the monthly interest on $10,000 would be $250, and so with monthly payments of $250 we would never pay off the loan. So the monthly interest rate must have been substantially under 2.5%. A bit of fooling around suggests taking $r_0 = 0.015$. We then find that $r_1 = 0.014411839$, $r_2 = 0.014394797$ and $r_3 = 0.01439477$. This suggests that to four significant figures the monthly interest rate is 1.439%.