Motivation for Chain Rule

We wish to motivate the formula

$$(f(g(x))' = f'(g(x))g'(x)$$

We first assert that

$$(f(cx+d))' = cf'(cx+d).$$

This follows from noting that the curve $y = f(cx+d)$ is the curve of $y = f(cx)$ shifted d units to the left. Then we note that the curve $y = f(cx)$ runs through the x values at a factor of c faster that does the curve $y = f(x)$ and hence the slopes are c times as big. You could verify this easily using the limit definition of derivative.

$$\lim_{h\to 0} \frac{f(c(x+h)+d) - f(cx+d)}{h} = \lim_{h\to 0} c\frac{f(cx+d+ch) - f(cx+d)}{ch}$$
$$= c\lim_{h\to 0} \frac{f(cx+d+ch) - f(cx+d)}{ch} = cf'(cx+d)$$

Consider a specific point x_0. We will verify/justify the chain rule at x_0; namely $(f(g(x))'$ at $x = x_0$ is

$$f'(g(x_0))g'(x_0).$$

Near x_0 we can approximate $g(x)$ by the linear approximation $mx+b$ where $m = g'(x_0)$ and b is chosen so that $mx_0 + b = g(x_0)$. Thus $g(x) \approx mx+b$ for x near x_0. Now we assert that $f(g(x)) \approx f(mx+b)$ (using the continuity of f, to be precise). We already note that $(f(mx+b))' = mf'(mx+b) = g'(x_0)f'(mx+b)$ and so $(f(g(x)))'$ at $x = x_0$ is approximately $g'(x_0)f'(g(x_0))$ (using $mx_0 + b = g(x_0)$). This is the Chain Rule!

Motivation for the Product Rule

We wish to motivate the Product Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

We use the same ideas as above. Consider a specific point x_0. We will verify/justify the product rule at x_0; namely $(f(x)g(x))'$ at $x = x_0$ is $f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Near x_0 we can approximate $f(x)$ by a linear function, the tangent line at x_0, say $m_f x + b_f$. We have $m_f = f'(x_0)$ and $f(x_0) = m_f x_0 + b_f$. Similarly we can approximate $g(x)$ by a linear function, the tangent line at x_0, say $m_g x + b_g$. We have $m_g = g'(x_0)$. and $g(x_0) = m_g x_0 + b_g$. Thus, for x near x_0 we have $f(x) \approx m_f x + b_f$ and $g(x) \approx m_g x + b_g$. Thus, for x near x_0 we have

$$f(x)g(x) \approx (m_f x + b_f)(m_g x + b_g) = m_f m_g x^2 + (m_f b_g + m_g b_f)x + b_f b_g.$$

Hence

$$\begin{array}{rcc} f(x)g(x))' \text{ at } x = x_0 & \approx & 2m_f m_g x_0 + (m_f b_g + m_g b_f) \\ & = & m_f(m_g x_0 + b_g) + m_g(m_f x_0 + b_f) \\ & = & f'(x_0)g(x_0) + g'(x_0)f(x_0) \end{array}$$

This is the product rule.

Neither of these motivations is a proof, but can be made into a proof using the formal definition for limits and derivatives.