Large Forbidden Configurations and Design Theory

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Design Theory

Definition Given an integer $m \geq 1$, let $[m] = \{1, 2, \ldots, m\}$.

Definition Given integers $k \leq m$, let $\binom{[m]}{k}$ denote all k- subsets of $[m]$.

Definition Given parameters t, m, k, λ, a t-(m, k, λ) design $\mathcal{D}$ is a multiset of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are exactly λ blocks $B \in \mathcal{D}$ containing S.

A t-(m, k, λ) design $\mathcal{D}$ is simple if $\mathcal{D}$ is a set (i.e. no repeated blocks).

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Definition Given parameters t, m, k, λ, a t-(m, k, λ) packing $\mathcal{P}$ is a set of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are at most λ blocks $B \in \mathcal{P}$ containing S.
(we will require a simple packing).

Theorem (Keevash 14) Let $1/m \ll \theta \ll 1/k \leq 1/(t+1)$ and $\theta \ll 1$. Suppose that $\binom{k-i}{t-i}$ divides $\binom{m-i}{t-i}$ for $0 \leq i \leq r-1$. Then there exists a t-(m,k,λ) simple design for $\lambda \leq \theta m^{k-t}$.

This covers a fraction θ of the possible range for $\lambda \in \left(0, \binom{m}{k}\binom{k}{t}/\binom{m}{t}\right)$.

Corollary (Weak Packing) Assume $0 < \alpha < k-t$. There exists a t-(m,k,m^α) packing $\mathcal{P}$ with $|\mathcal{P}|$ being $\Theta(m^{t+\alpha})$.

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

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$\text{Avoid}(m, F) = \{\, A \,:\, A \text{ is } m\text{-rowed simple, } F \nprec A\}$

$\text{forb}(m, F) = \max_A\{\|A\| : A \in \text{Avoid}(m, F)\}$

Let $s \cdot F$ denote $\overbrace{[F|F|\cdots|F]}^{s}$.

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We are interested in $\text{forb}(m, s \cdot F)$. An example:

$$\text{Let} \qquad F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let α be given. Then $\text{forb}(m, m^{\alpha} \cdot F)$ is $\Theta(m^{3+\alpha})$.

Let $s \cdot F$ denote $\overbrace{[F|F|\cdots|F]}^{s}$.

We consider $\text{forb}(m, s \cdot \begin{bmatrix}1\\1\end{bmatrix})$. Note that $s \cdot \begin{bmatrix}1\\1\end{bmatrix} = \overbrace{\begin{bmatrix}1\,1\cdots1\\1\,1\cdots1\end{bmatrix}}^{s}$

A pigeonhole argument yields

$$\text{forb}(m, s \cdot \begin{bmatrix}1\\1\end{bmatrix}) \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{s-2}{3}\binom{m}{2}.$$

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For fixed s, we have that $\text{forb}(m, s \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $O(m^2)$.

What happens for s that grows with m?

Weak Packing for $t = 2$: Let $\alpha > 0$ be given. There exist a constant $c_\alpha > 0$ so that

$$\text{forb}(m, m^\alpha \cdot \begin{bmatrix}1\\1\end{bmatrix}) \geq c_\alpha m^{2+\alpha}$$

i.e. $\text{forb}(m, m^\alpha \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $\Theta(m^{2+\alpha})$

Theorem $\mathrm{forb}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$.

Proof: We note that $[K_m^0 K_m^1 K_m^2 K_m^3] \in \mathrm{Avoid}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

Thus $\mathrm{forb}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) \geq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$.

(note that each pair of rows of has $(m-1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

We can argue, using the pigeonhole argument,

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and so $\text{forb}(m, m \cdot \begin{bmatrix}1\\1\end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$. ■

Thus $\text{forb}(m, m \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $\Theta(m^3)$.

Can we deduce the growth of $\text{forb}(m, m^{\alpha} \cdot \begin{bmatrix}1\\1\end{bmatrix})$?

Simple Triple Systems

Theorem (Dehon, 1983) Let m, λ be given. Assume $m \geq \lambda + 2$ and $m \equiv 1, 3 (\text{mod } 6)$. Then there exists a simple triple system, a simple $2-(m, 3, \lambda)$ design.

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Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple $2 - (m, 3, \lambda)$ design.

Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3}\binom{m}{2}$ simple matrix with all columns of column sum 3 and $T_{m,\lambda} \in \text{Avoid}(m, (\lambda + 1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$

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Thus, choosing $\lambda = m^{1/2} - 2$, we have
$\text{forb}(m, m^{1/2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ is $\Theta(m^{5/2})$
or more generally, $\text{forb}(m, m^{\alpha} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ is $\Theta(m^{2+\alpha})$ for $0 < \alpha \leq 1$.

Theorem
$\mathrm{forb}(m,(m+\binom{m-2}{2})\cdot\begin{bmatrix}1\\1\end{bmatrix})=\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\binom{m}{3}+\binom{m}{4}$.

Proof: Note $[K_m^0K_m^1K_m^2K_m^3K_m^4]\in\mathrm{Avoid}(m,(m+\binom{m-2}{2})\cdot\begin{bmatrix}1\\1\end{bmatrix})$.
Thus $\mathrm{forb}(m,(m+\binom{m-2}{2})\cdot\begin{bmatrix}1\\1\end{bmatrix})\geq\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\binom{m}{3}+\binom{m}{4}$.
We can argue for $s>m$, using the pigeonhole argument,

$$\mathrm{forb}(m,s\cdot\begin{bmatrix}1\\1\end{bmatrix})\leq\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\binom{m}{3}+\frac{s-m}{6}\binom{m}{2}$$

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$\text{forb}(m, (m + \binom{m-2}{2})) \cdot \begin{bmatrix}1\\1\end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \binom{m}{4}$.

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Thus $\text{forb}(m, (m + \binom{m-2}{2})) \cdot \begin{bmatrix}1\\1\end{bmatrix}) \geq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \binom{m}{4}$.
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Thus $\text{forb}(m, (m + \binom{m-2}{2})) \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $\Theta(m^4)$.

Let $\mathbf{1}_t$ denote the column of t 1's. The following result follows from Keevash 14.

Weak Packing: Let α and t be given. There exist a constant $c_{\alpha,t} > 0$ so that

$$\text{forb}(m, m^{\alpha} \cdot \mathbf{1}_t) \geq c_{\alpha,t} m^{t+\alpha}$$

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We form a matrix in $\text{Avoid}(m, m^{\alpha} \cdot \mathbf{1}_t)$ by first taking all columns up to some appropriate size, and then use the Weak Packing that follows as a Corollary to Keevash' design result.

Main Upper Bound Proof

Lemma Let F be a simple matrix and let $s > 1$ be given.
$\mathrm{forb}(m, s \cdot F) \leq \sum_{i=1}^{m-1} (s-1) \cdot \mathrm{forb}(m-i, F)$

Proof: We use the induction idea of A. and Lu 13.

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We will allow matrices to be non-simple in a restricted way

Allowing non-simple matrices

Let A be a (0,1)-matrix with $s \cdot F \nprec A$. Let $\mathbf{x}$ be a column of A.

Definition $\mu(\mathbf{x}, A)$=multiplicity of $\mathbf{x}$ as a column of A

Definition We say A is $(s-1)-$simple if $\mu(\mathbf{x}, A) \leq s-1 \qquad \forall \mathbf{x}$.

Assume A is $(s-1)-$simple

$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ G & H \end{array} \right]$$

If $\mu(\mathbf{y}, G) + \mu(\mathbf{y}, H) \geq s$, then set

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$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ G & H \end{array} \right] = \left[\begin{array}{cccc} 0\,0 & \cdots 0 & 1\,1 & \cdots 1 \\ B & C & C & D \end{array} \right]$$

If $\mu(\mathbf{y}, G) + \mu(\mathbf{y}, H) \geq s$, then set $\mu(\mathbf{y}, C) = \min\{\mu(\mathbf{y}, G), \mu(\mathbf{y}, H)\}$

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If $\mu(\mathbf{y}, G) + \mu(\mathbf{y}, H) \ge s$, then set $\mu(\mathbf{y}, C) = \min\{\mu(\mathbf{y}, G), \mu(\mathbf{y}, H)\}$
Then $[BCD]$ is $(s-1)-$simple.

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If $\mu(\mathbf{y}, G) + \mu(\mathbf{y}, H) \ge s$, then set $\mu(\mathbf{y}, C) = \min\{\mu(\mathbf{y}, G), \mu(\mathbf{y}, H)\}$
Then $[BCD]$ is $(s-1)-$simple.
Also $F \nprec C$ since each column $\mathbf{y}$ in C will appear s times in $[GH] = [BCD]$ and then $F \prec C$ will imply $s \cdot F \prec A$, a contradiction.

Main Upper Bound Proof

Lemma Let F be a simple matrix and let $s > 1$ be given.
$\text{forb}(m, s \cdot F) \leq \sum_{i=1}^{m-1} (s-1) \cdot \text{forb}(m-i, F)$.

Proof: (continued)

$$A = \left[\begin{array}{cccc} 0\,0 \cdots 0 & & 1\,1 \cdots 1 & \\ B & C & C & D \end{array} \right]$$

$F \not\prec C$ and so $\|C\| \leq (s-1) \cdot \text{forb}(m-1, F)$.
Now repeat on the $(m-1)-$rowed $(s-1)-$simple matrix BCD using

$$\text{forb}(m, s \cdot F) = \|A\| = \|[BCD]\| + \|C\| \quad \blacksquare$$

$$\text{Let } F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We have $\text{forb}(m, F) = 4m$,

i.e. $\text{forb}(m, F)$ is $O(m)$.

Theorem Let $\alpha > 0$ be given. Using the Weak Packing, $\text{forb}(m, m^{\alpha} \cdot F)$ is $\Theta(m^{2+\alpha})$.

Proof:

$\text{forb}(m, m^{\alpha} \cdot F) \leq \sum_{i=1}^{m-1} m^{\alpha} \cdot \text{forb}(m-i, F) = m^{\alpha} \sum_{i=1}^{m-1} 4(m-i)$.

Now $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \prec F$ and so $m^{\alpha} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \prec m^{\alpha} \cdot F$ from which we have

$\text{forb}(m, m^{\alpha} \cdot F) \geq \text{forb}(m, m^{\alpha} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$. ■

$$\text{Let} \quad F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\text{forb}(m, F)$ is $O(m^2)$. As before $s \cdot \mathbf{1}_3 \prec s \cdot F$ and so $\text{forb}(m, s \cdot F) \geq \text{forb}(m, s \cdot \mathbf{1}_3)$.

Theorem Let $\alpha > 0$ be given. Using the Weak Packing, $\text{forb}(m, m^{\alpha} \cdot F)$ is $\Theta(m^{3+\alpha})$.

There are a number of F which yield nice results assuming the Weak Packing. There are cases which do not yield the desired results.

$$\text{Let} \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem (Frankl, Füredi, Pach 87) $\text{forb}(m, F) = \binom{m}{2} + 2m - 1$ i.e. $\text{forb}(m, F)$ is $O(m^2)$.

Theorem (A. and Lu 13) Let s be given. Then $\text{forb}(m, s \cdot F)$ is $\Theta(m^2)$.

Conjecture $\text{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{2+\alpha})$.

We can only prove that $\text{forb}(m, m^\alpha \cdot F)$ is $O(m^{3+\alpha})$.

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