UBC Mathematics Circle January 16, 2017 Richard Anstee

Induction

We are often asked to prove results which are of the form "$H(n)$ is true for all $n \geq 1$", where $H(n)$ is some statement that refers to n.

To establish "$H(n)$ is true for all $n \geq 1$", we apparently have an infinite amount of work. There are an infinite number of choices for n. So we use facts about integers to reduce the workload. Think of an infinite ladder with the steps on the ladder labelled step 1, step 2, step 3,$\cdots$. We imagine reaching step n is the same as having proved $H(n)$. Now imagine we can establish the following facts

$H(1)$ is true	Base Case
$H(k) \implies H(k+1)$ for all $k \geq 1$	Inductive Step

In ladder terms this becomes

we can reach step 1	Base Case
if we can reach the kth step then we can reach the $(k+1)$st step	Inductive Step

It should be reasonable that if these two hypotheses are true then we can reach any step of the ladder. Thus $H(n)$ is true for all n.

There are many variations in how induction is applied. One possibility is that you don't start ot the first step 1 but at some later step, say step 3. Now if we can reach step 3 and, for all $k \geq 3$, if we can reach the kth step then we can reach the $(k+1)$st step then we can reach any step n in the ladder for $n \geq 3$.

Another variation that is widely used is to replace $H(n)$ by $H'(n)$ where $H'(n)$ is the statement that $H(k)$ is true for all k with $1 \leq k \leq n$. The statement $H'(1)$ is true is just the statement $H(1)$ is true. The statement $H'(k) \implies H'(k+1)$ is equivalent to showing that $H(1), H(2), \ldots, H(k)$ are true imples $H(k+1)$ is true. And the conclusion that $H'(n)$ is true for all $n \geq 1$ is eqiuivalent to $H(n)$ is true for all n. While this is just induction with an altered induction hypothesis it is sometimes called *strong* induction.

For example:

We are familiar with the fibonaci numbers
$f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 8, f(7) = 13, \ldots$ which are generated by a recurrence $f(n) = f(n-1) + f(n-2)$ starting with the base case that $f(1) = f(2) = 1$. We would like to say that they grow exponentially. A possible choice for $H(n)$ is the statement that there exists a constant c, independent of n, so that

$$H(n): \quad \text{The } n\text{th fibonacci number is greater that } c \cdot (1.5)^n$$

The growth rate $(1.5)^n$ is just a guess. We have the recuurence for fibonacci numbers that

$$f(n) = f(n-1) + f(n-2)$$

and so we would like to use properties of both $f(n-1)$ and $f(n-2)$ to determine properties of $f(n)$. We will need to prove $H(n-1), H(n-2) \implies H(n)$, a form of strong induction. As well we must establish the base case $H(1)$ and $H(2)$.

Problems

Eden Sequences

Adapted from 2012 Euclid Contest, problem 10. An Eden sequence chosen from $\{1, 2, \ldots, N\}$ is a sequence $a_1, a_2, \ldots$ of integers from $\{1, 2, \ldots, N\}$ with the properties that the sequence is increasing and the terms in the sequence in even position are even and the terms in odd position are odd. One can verify that the following are the four Eden sequences chosen from $\{1, 2, 3\}$:

$$1 \qquad 1,2 \qquad 3 \qquad 1,2,3$$

Let $e(N)$ denote the number of Eden sequences from $\{1, 2, \ldots, N\}$. Determine a recurrence for $e(N)$ in terms of $e(k)$ for some $k < N$. The original problem asked you to compute $e(18)$ and $e(19)$ given that $e(17) = 4180$ and $e(20) = 17710$.

Hint: You may find it easier to consider $f(k)$ to be the number of Eden sequences whose largest term is k.

Extra credit: Why was the name Eden sequence chosen?

Towers of Hanoi

Consider the following wooden puzzle consisting of n disks with the ith disk having radius i. Imagine you have a board with three pegs and the disks have holes at their centres so that they can be placed on the pegs. A legal placement of the disks has no larger disk lying on top of a smaller disk. A move is moving one disk on the top of a peg to the top of another peg. All the disks are initially places on one peg, from largest to smallest. How many moves are required to move all the disks to another peg?

Recurrences from Computer Science

Imagine we have a function f that when evaluated at n yields $f(n)$ is the runtime of the algorithm on a problem of size n. Now assume $f(1) = 1$ and for $n \geq 2$,

$$f(n) \leq f(\left\lfloor \frac{n}{2} \right\rfloor) + f(\left\lceil \frac{n}{2} \right\rceil) + 1$$

Verify that for some constant c that $f(n) \leq cn$ for all $n \geq 1$.

Convex n-gons

An n-gon is *convex* if the interior angle at any corner is at most π. Or if we go around the n-gon counterclockwise we will always be turning to the left. Consider a convex n-gon with n vertices/corners. Show that the sum of the interior angles is $(n-2)\pi$ or in degrees $180(n-2)$. Does this generailize? Hint: Try $n = 3$.

Lines

Consider n lines in the plane. Consider the regions formed in the plane by the lines. You can think of this as cutting the plane along each line and the regions are the connected pieces. Some regions will be infinite. Show that you can colour each region either black or white so that no two regions bordering each other, on a non-zero length of one of the lines, have the same colour.

Proof: For the base case we need to establish $H(1), H(2)$. Since we need $f(1) = 1 \leq c(1.5)$ and $f(2) = 1 \leq c(1.5)^2$, we can take $c = 4/9$ and then $H(1), H(2)$ are true.

For the inductive step we need to show that $H(k-1), H(k) \implies H(k+1)$ for $k \geq 3$. We check

$$f(k+1) = f(k) + f(k-1) \geq c(1.5)^k + c(1.5)^{k-1} = c(1.5)^{k+1}\left(\frac{1}{1.5} + \frac{1}{(1.5)^2}\right)$$

We verify that $\frac{1}{1.5} + \frac{1}{(1.5)^2} = \frac{2}{3} + \frac{4}{9} = \frac{10}{9} > 1$ and so we deduce that $f(k+1) \geq c(1.5)^{k+1}$ establishing $H(k+1)$. By induction, $H(n)$ is true for all $n \geq 1$. ■

Can we choose a larger number that 1.5 that still works? (We may need a smaller c). In fact the largest number that still works is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ so that for a suitable c, $f(n) \geq c\tau^n$. You can then go on to establish that there exists a constant c' so that $f(n) \leq c'\tau^n$.