Forbidden Configurations: Quadratic Bounds

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Abstract

A *simple* matrix is a $\{0,1\}$-matrix with no repeated columns. For a $\{0,1\}$-matrix F, define $F \prec A$ if there is a submatrix of A which is a row and column permutation of F. Let $\|A\|$ denote the number of columns of A. Define

$$\text{forb}(m, F) = \max\{\|A\| : A \text{ is } m\text{-rowed simple matrix and } F \not\prec A\}.$$

We classify all 6-rowed configurations F for which forb(m, F) is $\Theta(m^2)$ and prove forb(m, F) is $\Omega(m^3)$ for all other 6-rowed F. We also prove forb(m, G) is $O(m^2)$ for a particular 5×6 simple G and the addition of any column α to G makes forb$(m, [G\,\alpha])$ to be $\Omega(m^3)$. The results are evidence for a conjecture of Anstee and Sali which predicts the asymptotics of forb(m, F) as a function of F.

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1 Introduction

The paper considers an extremal problem. Some of the most celebrated extremal results are those of Erdős and Stone [ES46] and Erdős and Simonovits [ES66]. They consider the following problem: Given $m \in \mathbb{N}$ and a graph F, find the maximum number of edges in a graph G on m vertices that avoids having a subgraph isomorphic to F. There are a number of ways to generalize this to hypergraphs. A k-uniform hypergraph is one in which each edge has size k. Some view k-uniform hypergraphs as the most natural generalization of a graph (a graph is a 2-uniform hypergraph) and one might also generalize the forbidden subgraph to a forbidden k-uniform subhypergraph. There are both asymptotic results e.g. Turán's problem and exact bounds e.g. [dCF00], [Pik08], [Für91]. We have generalized in a different (but also natural) way. We consider the following. Given $m \in \mathbb{N}$ and a hypergraph F, find the maximum number of edges in a simple hypergraph H on m vertices that avoids having a subhypergraph isomorphic to F. We find the language of matrices convenient.

Define a matrix to be *simple* if it is a $\{0,1\}$-matrix with no repeated columns. Then an $m \times n$ simple matrix corresponds to a *simple hypergraph* or *set system* on m vertices with n edges. Let $\|A\|$ denote the number of columns in A (which is the cardinality of the associated set system). Our objects of study are $\{0,1\}$-matrices with row and column order information stripped from them. Define two $\{0,1\}$-matrices to be *equivalent* if one is a row and column permutation of another. This defines an equivalence relation. A representative of each equivalence class is called a *configuration*. Abusing notation, we will commonly use matrices and their corresponding configurations interchangeably.

Definition 1.1. *For a configuration F and a $\{0,1\}$-matrix A (or a configuration A), we say that F is a* subconfiguration *of A, and write $F \prec A$ if there is a representative of F which is a submatrix of A. We say A* has no configuration F *(or* doesn't contain F as a configuration*) if F is not a subconfiguration of A. Let* $\text{Avoid}(m, F)$ *denote the set of all m-rowed simple matrices with no configuration F.*

Our main extremal problem is to compute

$$\text{forb}(m, F) = \max_A \{\|A\| \ : \ A \in \text{Avoid}(m, F)\}.$$

A survey on the topic can be found in [Ans]. Let A^c denote the $\{0,1\}$-complement of A (replace every 0 in A by a 1 and every 1 by a 0). Note that $\text{forb}(m, F) = \text{forb}(m, F^c)$.

Remark 1.2. *Let F and G be configurations such that $F \prec G$. Then* $\text{forb}(m, F) \leq \text{forb}(m, G)$.

We will also consider families of forbidden configurations: Let $\mathcal{F} = \{F_1, F_2, \ldots, F_s\}$ be a set of configurations. We define $\text{Avoid}(m, \{F_1, F_2, \ldots, F_s\})$ to be the set of all m-rowed simple configurations A for which $F_i \not\prec A$ for all $i \in \{1, 2, .., s\}$. This yields the extremal problem

$$\text{forb}(m, \{F_1, F_2 ..., F_s\}) = \max_A \{\|A\| : A \in \text{Avoid}(m, \{F_1, F_2, ..., F_s\})\}.$$

For two given $\{0,1\}$-matrices A, B which have the same number of rows, let $[A\,|\,B]$ denote the matrix of A concatenated with B. Note that this is not a well defined operation on configurations but we find it convenient and unambiguous in our paper. We use it on representatives of a configuration where it is well defined. For a set of rows S, we let $A|_S$ denote the submatrix of A given by the rows S. We say a column α has *column sum* t if it has exactly t ones. Define $\mathbf{0}_m$ to be a column with m 0's and $\mathbf{1}_m$ to be a column of m 1's.

An important general result is due to Füredi.

Theorem 1.3. *[Für83] Let F be a given k-rowed $\{0,1\}-$matrix. Then* $\text{forb}(m, F)$ *is* $O(m^k)$.

We desire more accurate asymptotic bounds. Anstee and Sali conjectured that the best asymptotic bounds can be achieved with certain product constructions.

Definition 1.4. *Let A and B be matrices. We define the* product $A \times B$ *by taking each column of A and putting it on top of every column of B. Hence if A, B are simple and $\|A\| = a$ and $\|B\| = b$ then $A \times B$ is simple with $\|A \times B\| = ab$. If A has c rows and B has d rows then $A \times B$ has $c + d$ rows.*

We are interested in asymptotic bounds for $\text{forb}(m, F)$. Let I_m be the $m \times m$ identity matrix, I_m^c be the $\{0,1\}$-complement of I_m (all ones except for the diagonal) and let T_m be the triangular matrix, namely the $\{0,1\}$-matrix with a 1 in position i, j if and only if $i \leq j$. Anstee and Sali conjectured that the asymptotically "best" constructions avoiding a single configuration would be products of I, I^c and T.

Conjecture 1.5. *[AS05] Let F is a configuration. Define $X(F)$ to be the largest number p such that for some choices $R_i \in \{I_r, I_r^c, T_r\}$ (for all sufficiently large r)*

$$F \nprec R_1 \times R_2 \times ... \times R_p.$$

Then

$$\text{forb}(m, F) = \Theta(m^{X(F)}).$$

Taking $m = r \cdot p$, the construction $R_1 \times ... \times R_p$ is an m-rowed matrix with $(m/p)^p = \Omega(m^p)$ columns avoiding F. Thus the fact that $\text{forb}(m, F)$ is $\Omega(m^{X(F)})$ is built into the conjecture. Proving the conjecture reduces to showing that $\text{forb}(m, F) = O(m^{X(F)})$. Note that the conjecture is silent on forbidden families of configurations. Because of Remark 1.2, we are particularly interested in *boundary cases*, which are configurations F for which the conjecture predicts $\text{forb}(m, F)$ is $\Theta(m^k)$, but for any column α either not appearing in F or appearing at most once, the product constructions give that $\text{forb}(m, [F|\alpha])$ is $\Omega(m^{k+1})$. Proving that F is a boundary case not only supports the conjecture but also helps in classifying all matrices F by the asymptotics of $\text{forb}(m, F)$.

The conjecture has been proven for all $k \times \ell$ configurations F with $k = 1, 2, 3$ and many others cases in various papers. The proofs for $k = 2$ are in [AGS97], for $k = 3$ in

[AGS97], [AFS01], [AS05]. For $\ell = 2$, the conjecture was verified in [AK06]. For $k = 4$, all cases either when the conjecture predicts a cubic bound for F or when F is simple were completed in [AF10]. For $k = 4$ and F non-simple, there are three boundary cases with quadratic bounds, one of which is established in [ARS10]. The following theorem classifies all 6-rowed configurations F for which $\text{forb}(m, F)$ is $\Theta(m^2)$ by giving the unique boundary case.

Theorem 1.6. *Let F be any 6-rowed configuration. Then* $\text{forb}(m, F)$ *is* $\Theta(m^2)$ *if and only if F is a configuration in*

$$G_{6\times3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, if $F \not\prec G_{6\times3}$, then $\text{forb}(m, F)$ *is* $\Omega(m^3)$.

We note that $G^c_{6\times3} = G_{6\times3}$ which is required by (1.2) and Theorem 1.6. Anstee and Keevash [AK06] established the asymptotic bounds for all $k \times 2$ configurations and in particular concluded that

$$\text{forb}\left(m, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \text{ and } \text{forb}\left(m, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \text{ are both } \Theta(m^2).$$

The proof of the second of these begins to use the full power of the proof in [AK06] and so it is interesting that Theorem 1.6 provides a generalization for both of them using an inductive proof (admittedly rather complicated using Theorem 1.7) that is quite different than that in [AK06].

In order to prove Theorem 1.6, we will use three results. First, Lemma 2.1 is the "only if" part of the theorem. The second, Lemma 2.2, generalizes Lemma 3.2 in [AK06]. Lastly, we will use the second main result in this paper, Theorem 1.7, which is of great interest on its own. Previous work of Chris Ryan, reported in [Ans], computed nine 5-rowed simple matrices F which by Conjecture 1.5 should be boundary cases and for which $\text{forb}(m, F)$ should be $\Theta(m^2)$. One of them, named F_7 in [Ans], is

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Note that $F_7 = F_7^c$.

Theorem 1.7. *We have that* $\text{forb}(m, F_7)$ *is* $\Theta(m^2)$. *Moreover, for any* 5×1 $\{0,1\}$-*column* α, $\text{forb}(m, [F_7 \,|\, \alpha])$ *is* $\Omega(m^3)$.

The proof uses Standard Induction (Section 3) and the linear bound of Lemma 3.1 (for three smaller matrices) which in turn uses Standard Induction (in a novel way). We give the proof of Theorem 1.7 from Lemma 3.1 in Section 3 and the proof of Lemma 3.1 in Section 5.

2 Classifying 6-rowed configurations for which forb is quadratic

Lemma 2.1. *Let* F *be a 6-rowed configuration such that* $F \not\prec G_{6\times 3}$. *Then* $\text{forb}(m, F)$ *must be* $\Omega(m^3)$.

Proof: We may assume all of F's columns have column sum 3, otherwise, if F had a column of column sum 4 or more, then $F \not\prec I \times I \times I$, and if F had a column sum of 2 or less, then $F \not\prec I^c \times I^c \times I^c$.

Without loss of generality, let the first column of F be $(1,1,1,0,0,0)^T$. With these assumptions, there are only a few cases left to check, and an exhaustive computer search revealed the lemma to be true. But we give here an explicit proof, if for no other reason than to check the computer code.

Note that the following 2-columned matrices have at least a cubic bound:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \not\prec I \times I \times I, \qquad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \not\prec I \times I \times T.$$

This means that to form F, we must put together columns of sum 3 such that for each pair of columns, the number of rows where both columns have 1's is either one or two. Here are all the possibilities for (the first) two columns having 1's in (the first) two rows in common:

$$\underset{\not\prec I^c \times I^c \times I^c}{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]}, \quad \underset{\not\prec I \times I \times I}{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}, \quad \underset{= G_{6\times 3}}{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]}, \quad \underset{\not\prec I^c \times I^c \times I^c}{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]}, \quad \underset{\not\prec I \times I \times I}{\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]}.$$

The only other possibility is that each pair of columns has a 1 in only one row in common.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\prec I \times I \times T.$$

Thus, the only four-columned matrices F for which $\text{forb}(m, F)$ could be $O(m^2)$ have to contain $G_{6\times3}$ in every three-columned subset. The only possibility is then

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \not\prec I \times I \times T\,,$$

which means $\text{forb}(m, F)$ is $\Omega(m^3)$. This concludes the lemma. ■

The following lemma generalizes Lemma 3.2 in [AK06].

Lemma 2.2. *Let*

$$F = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ & F' & \end{bmatrix}.$$

Then we can conclude that

$$\text{forb}(m, F) \leq \text{forb}\left(m, \begin{bmatrix} 1 & \cdots & 1 \\ & F' & \end{bmatrix}\right) + \text{forb}\left(m, \begin{bmatrix} 0 & \cdots & 0 \\ & F' & \end{bmatrix}\right). \tag{2.1}$$

Proof: Let $A \in \text{Avoid}(m, F)$ with $\|A\| = \text{forb}(m, F)$. Then permute the columns of A (take another representative in the equivalence class) and write it as

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & A' & & & A'' & \end{bmatrix}.$$

Note that A' and A'' are simple. Since A' cannot have $\begin{bmatrix} 1 & \cdots & 1 \\ & F' & \end{bmatrix}$ as a subconfiguration, and A'' cannot have $\begin{bmatrix} 0 & \cdots & 0 \\ & F' & \end{bmatrix}$ as a subconfiguration, the bound (2.1) follows. ■

From the previous lemma, we note that $G_{6\times3}$ has a row of 0's and a row of 1's, and therefore the quadratic bound for $\text{forb}(m, G_{6\times3})$ would follow from quadratic bounds for forb(m,G) and forb(m,G'), with G and G' obtained by removing the row of 1's and the row of 0's from $G_{6\times3}$ respectively:

$$G=\begin{bmatrix}1&1&0\\1&0&1\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}\quad\text{and}\quad G'=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\0&1&0\\0&0&1\end{bmatrix}.$$

We will prove more, as both are contained in the boundary case F_7. Observe that $G'=G^c$ as configurations. We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6: To prove forb$(m,G_{6\times3})$ is $O(m^2)$ we use Lemma 2.2. We check that $G\prec F_7$ and $G'\prec F_7$. Then (2.1) yields forb$(m,G_{6\times3})\le$ forb$(m,G)+$forb$(m,G')\le$ 2forb(m,F_7). Now Theorem 1.7 shows that forb(m,F_7) is $O(m^2)$ which then implies forb$(m,G_{6\times3})$ is $O(m^2)$. Lemma 2.1 verifies that every configuration F not contained in $G_{6\times3}$ has forb(m,F) being $\Omega(m^3)$. ∎

We need only prove Theorem 1.7, which forms the rest of the paper.

3 Standard Induction

In this section we consider the Standard Induction argument [Ans]. Let F be a configuration and suppose we have $A\in$ Avoid(m,F). Consider deleting a row r. The resulting matrix might not be simple. Let C_r be the simple matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. For example, if we permute the rows and columns of A so that r becomes the first row, then after some column permutations we obtain the *standard decomposition* of A as follows:

$$A=\begin{matrix}r\rightarrow\\ \\\end{matrix}\begin{bmatrix}0&\cdots&0&1&\cdots&1\\B_r&&C_r&C_r&&D_r\end{bmatrix},\tag{3.1}$$

where B_r are the columns that appear with a 0 on row r, but don't appear with a 1, and D_r are the columns that appear with a 1 but not a 0. We note $[B_rC_rD_r]$ is a simple $(m-1)$-rowed matrix avoiding F. If we assume $\|A\|=$ forb(m,F), then we obtain

$$\|A\|=\text{forb}(m,F)\le\|C_r\|+\text{forb}(m-1,F).\tag{3.2}$$

This means any upper bound on $\|C_r\|$ (as a function of m), automatically yields an upper bound on forb(m,F) by induction. If we remove any row from F and call the resulting configuration F' then

$$F\prec\left[\begin{array}{cc}00\cdot0&11\cdots1\\F'&F'\end{array}\right].$$

Thus C_r can't have F' as a configuration since C_r is exactly the set of columns that appear with both a 0 and a 1 in row r. We can search for a row r such that $\|C_r\|$ is as small as possible. If we can prove that there is a row r with $\|C_r\|$ small enough, we can proceed then by induction using (3.2). We now describe how to apply Standard Induction to prove the quadratic bound for $\mathrm{forb}(m, F_7)$ by proving a linear bound for $\|C_r\|$.

Let $A \in \mathrm{Avoid}(m, F_7)$ and apply the *standard decomposition* of (3.1) for $r = 1$. Our goal is to show $\|A\|$ is quadratic by showing that $\|C_1\|$ is linear. We note that C_1 cannot contain any of the configurations H_1, H_2, H_3, H_4, H_5:

$$H_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ H_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ H_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ H_5 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

We observe that $H_3^c = H_3$, $H_4 = H_1^c$, $H_2^c = H_5$. Also $H_3 \prec H_1$ (columns 2,3,5,6) and $H_3 \prec H_4$ and so we may ignore H_1, H_4. We state a lemma we need in order to prove Theorem 1.7.

Lemma 3.1. *We have that* $\mathrm{forb}(m, \{H_2, H_3, H_5\})$ *is* $O(m)$.

We will prove Lemma 3.1 in the Section 5. We can now prove that $\mathrm{forb}(m, F_7)$ is quadratic.

Proof of Theorem 1.7: The fact that $\mathrm{forb}(m, F_7)$ is $\Omega(m^2)$ comes directly out of the conjecture, as $F_7 \not\prec I \times I$. We show $\mathrm{forb}(m, F_7)$ is $O(m^2)$ using induction on m. Consider $A \in \mathrm{Avoid}(m, F_7)$ with $\|A\| = \mathrm{forb}(m, F_7)$. Then using (3.2), we have

$$\mathrm{forb}(m, F_7) = \|A\| \leq \mathrm{forb}(m-1, \{H_2, H_3, H_5\}) + \mathrm{forb}(m-1, F_7).$$

Given that there is a constant c so that $\mathrm{forb}(m-1, \{H_2, H_3, H_5\}) \leq c(m-1)$ by Lemma 3.1, we deduce the quadratic bound for $\mathrm{forb}(m, F_7)$.

Now consider any 5×1 column α. We deduce that $\mathrm{forb}(m, [F_7 \,|\, \alpha])$ is $\Omega(m^3)$ for α having zero, one, four or five 1's, or if α is a column in F_7 (considered as a matrix). It is a computational exercise to show that every other α results in $\mathrm{forb}(m, [F_7 \,|\, \alpha])$ being $\Omega(m^3)$. We need only consider α having two 1's since $F_7^c = F_7$. If α has 0's on rows 2,3 then $[F_7 \,|\, \alpha] \not\prec I^c \times I^c \times I^c$ (each pair of rows from the four rows 1,2,3,4 of $[F \,|\, \alpha]$ has $(0,0)^T$) or two 0's on rows 1,4 then $[F_7 \,|\, \alpha] \not\prec I^c \times I^c \times I^c$ (each pair of rows from the four rows 1,3,4,5 has $(0,0)^T$) . This only leaves $\alpha = (0,0,1,1,0)^T$ (the other three choices are in F_7) and in such case $[F_7 \,|\, \alpha] \not\prec T \times T \times T$ since every pair of rows from the four rows 1,2,3,4 has the 2×2 configuration I_2. ■

4 What is Missing?

In this section we study another tool that has been extensively used in Forbidden Configurations. For lack of a better name, the tool is named "What is Missing if a family of configurations $\mathcal{F}$ is avoided?", or for short, "What is Missing?". This technique works for general configurations but in this paper we only need it for simple configurations. Let F be a simple configuration. Let $A \in \text{Avoid}(m, F)$. For some $s \in \mathbb{N}$ (typically s is the number of rows of F), consider all s-tuples of rows from A and for each s-tuple of rows S, consider the matrix $A|_S$ formed from rows S of A. For example, if $S = \{2, 3, 4\}$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{then} \quad A|_S = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.1}$$

Without any restriction, $A|_S$ could have all 2^s possible columns (each appearing multiple times perhaps). But we have the restriction that $F \nprec A$, so in particular $F \nprec A|_S$, so some of the columns have to be missing. For the example, in $A|_S$, the columns $[0, 0, 0]^T$ and $[1, 0, 0]^T$ appear twice, while $[1, 1, 0]^T$, $[1, 0, 1]^T$ and $[1, 1, 1]^T$ appear once, but $[0, 1, 0]^T$, $[0, 0, 1]^T$ and $[0, 1, 1]^T$ don't appear at all.

For an s-tuple of rows we say a column (of size s) is *absent* or *missing* if it doesn't appear. We say it is *present* if it does. We search for the various possibilities of which columns are missing for every s-tuple when forbidding F. For example, suppose

$$F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume $A \in \text{Avoid}(m, F)$. Then for every triple of rows (a, b, c) of A, there is an ordering (i, j, k) of (a, b, c), for which the columns marked by *no* are absent satisfy are in one of the following four cases:

$$\begin{matrix} & \text{no} & \text{no} & \text{no} \\ i & 1 & 0 & 0 \\ j & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{matrix} \quad \text{or} \quad \begin{matrix} & \text{no} & \text{no} & \text{no} \\ i & 1 & 1 & 0 \\ j & 1 & 0 & 1 \\ k & 0 & 1 & 1 \end{matrix} \quad \text{or} \quad \begin{matrix} & \text{no} & \text{no} & \text{no} \\ i & 1 & 0 & 0 \\ j & 0 & 1 & 0 \\ k & 1 & 1 & 1 \end{matrix} \quad \text{or} \quad \begin{matrix} & \text{no} & \text{no} & \text{no} \\ i & 1 & 0 & 1 \\ j & 0 & 1 & 1 \\ k & 0 & 0 & 0 \end{matrix}. \tag{4.2}$$

Of course if there are no columns of column sum 1 or if there are no columns of column sum 2 in $A|_S$ (the first two cases), then $F \nprec A|_S$. The third and fourth examples might be harder to see, but if we take a look at the columns that could appear, we see why:

$$\overset{\text{absent}}{\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}} \quad \Longrightarrow \quad \overset{\text{possibly present}}{\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}},$$

$$\overset{\text{absent}}{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}} \quad\Longrightarrow\quad \overset{\text{possibly present}}{\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}\ .$$

We note that F doesn't appear in the present columns in either case. For example the matrix A of (4.1) avoids F and for $S = \{2, 3, 4\}$ we find the rows are in the fourth case of (4.2) with $i = 3$, $j = 4$ and $k = 2$.

We wrote a C++ program whose input is a configuration F (or a family of configurations $\mathcal{F}$), and its output is the list of possibilities for columns absent. Studying this list is often easier than studying F for the purpose of analyzing the structure of a matrix that doesn't have F as a configuration. Unfortunately, the program performs $O(2^{2^s})$ configuration comparison operations. In practice, this means checking configurations with $s \le 4$ is almost instantaneous, $s = 5$ takes, depending on the configuration, anywhere from a few minutes to a couple of hours, and with $s = 6$ it's typically hopeless.

Applying the above technique to $\mathcal{F} = \{H_2, H_3, H_5\}$, we get the following lemma.

Lemma 4.1. *Let $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$. Then there are 13 possibilities Q_0, Q_1, $\ldots, Q_{12}$ for what is missing on each 4-set of rows:*

$$Q_0 = \overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}}, \quad Q_1 = \overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}},$$

$$Q_2 = \overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}, \quad Q_3 = \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}},$$

$$Q_4 = \overset{\text{no}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}, \quad Q_5 = \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}},$$

$$Q_6 = \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}, \quad Q_7 = \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}\overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}},$$

$$Q_8 = \overset{\text{no}}{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \quad , \quad Q_9 = \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}} \quad ,$$

$$Q_{10} = \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \quad , \quad Q_{11} = \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}} \quad ,$$

$$Q_{12} = \overset{\text{no}}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \overset{\text{no}}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}} \quad .$$

Proof of Lemma 4.1: An exhaustive computer search yields the result. ■

5 Linear bound for $\text{forb}(m, \{H_2, H_3, H_5\})$

The rest of the paper is a proof of Lemma 3.1. Let $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$. We will use special features of H_2, H_3, H_5 to obtain a linear bound on $\|A\|$. The forbidden configuration H_3 is used most often in this proof. We will show $\|A\| \leq 7m$ by induction on m. We analyze the 13 cases of Lemma 4.1 one by one and have special arguments for the three troublesome cases Q_2, Q_3, Q_{11}.

Lemma 5.1. *Let $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$. Consider the standard decomposition (3.1) of A based on row r. Let $L(r) \neq \emptyset$ be a minimal set of rows such that $C_r|_{L(r)}$ is simple. Then each triple of rows $\{i, j, k\}$ in $L(r)$ yield a quadruple of rows $\{r, i, j, k\}$ on which one of the cases Q_2, Q_3, Q_{11} occurs, with row r being the first row of each of the cases Q_2, Q_3, Q_{11} as given in Lemma 4.1.*

Proof: Define K_k as the unique $k \times 2^k$ simple configuration consisting of all possible columns on k rows. For each Q_i we record pairs of rows containing "a copy of K_2": namely in the columns marked absent we find

$$\begin{matrix} r \\ i \\ j \\ k \end{matrix} \overset{\text{no}}{\begin{bmatrix} a \\ e \\ 0 \\ 0 \end{bmatrix}}, \overset{\text{no}}{\begin{bmatrix} b \\ f \\ 1 \\ 0 \end{bmatrix}}, \overset{\text{no}}{\begin{bmatrix} c \\ g \\ 0 \\ 1 \end{bmatrix}}, \overset{\text{no}}{\begin{bmatrix} d \\ h \\ 1 \\ 1 \end{bmatrix}}.$$

Suppose A had these columns missing on the quadruple of rows r, i, j, k and that rows i, j, k belong to $L(r)$. Then in the simple matrix C_r from (3.1) has the four 3×1 columns $(e, 0, 0)^T$, $(f, 1, 0)^T$, $(g, 0, 1)^T$ and $(h, 1, 1)^T$ missing on the triple of rows $\{i, j, k\}$. We deduce that row i cannot belong to $L(r)$, a contradiction.

By analyzing the cases, we find that $Q_0, Q_1, Q_5, Q_6, Q_7, Q_8, Q_{10}, Q_{12}$ have 3 rows each pair of which have a "K_2" and Q_4, Q_9 have two disjoint pairs of rows each with a "K_2". Thus in any of these cases, what is missing on a triple of rows in C_r will contain a copy of "K_2" and so we can delete a row from C_r without disturbing simplicity of the remainder of C_r. In cases Q_2, Q_3, Q_{11}, if we choose row r to be any row but the first row in each of the cases then there is a "K_2" on the remaining triple. ■

We would like to show that for all $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$ we can choose row r so that $\|C_r\| \leq 7$ as in 3.1. Then by (3.2) and induction, $\|A\| \leq 7m$. We will assume the contrary, namely that there is $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$ such that for every row r, $\|C_r\| \geq 8$.

In each of the troublesome cases Q_2, Q_3, Q_{11}, we end up with the following sets of columns missing on a triple of rows in C_r (arising from what is missing in A on a quadruple of rows involving r) and we name the cases correspondingly P_2, P_3, P_{11}.

$$P_2 : \quad \overset{\text{no}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \tag{5.1}$$

$$P_3 : \quad \begin{matrix} i \\ j \\ k \end{matrix} \overset{\text{no}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \text{ yielding } \begin{matrix} i \\ j \\ k \end{matrix} \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}} \tag{5.2}$$

$$P_{11} : \quad \overset{\text{no}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \quad \overset{\text{no}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \tag{5.3}$$

Lemma 5.2. *Let $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$. Consider the standard decomposition (3.1) of A based on row r. Let $L(r) \neq \emptyset$ be a minimal set of rows such that $C_r|_{L(r)}$ is simple. Then each triple of rows $\{i, j, k\}$ in $L(r)$ is in one of the cases P_2, P_3 or P_{11}. Moreover, if any triple in $L(r)$ is in case P_2, then all triples of rows of $L(r)$ are in case P_2. Similarly if any triple in $L(r)$ is in case P_3 (respectively P_{11}), then all triples of rows are in case P_3 (resp. P_{11}).*

Proof: By Lemma 5.1, every triple of rows of $L(r)$ satisfies one of P_2, P_3 or P_{11}. A triple of rows $\{a, b, c\}$ in case P_3 can't overlap with a triple of rows in case P_2 (respectively P_{11}) on two rows $\{a, b\}$ since on the two rows $\{a, b\}$ what is missing (by 5.2) will extend to one new column missing on the triple from P_2 (resp. P_{11}) yielding a "K_2". This would allow us to delete a further row from $C_r|_{L(r)}$ while preserving simplicity, a contradiction to the fact that $L(r)$ is minimal with $C_r|_{L(r)}$ simple. Thus, if any triple of rows of $L(r)$ is in case P_3, then all triples of rows of $L(r)$ are in case P_3. Assume all triples of rows are in case P_2 or P_{11}.

We can't have a triple of rows in case P_2 overlap with a triple of rows in case P_{11} on two rows as shown below. On the quadruple of rows we have marked 'OK' over the columns which can occur on the quadruple of rows. At most 6 columns can be present in $C_r|_{L(r)}$ and we note that we can delete the second or third row from $C_r|_{L(r)}$ and not affect simplicity of $C_r|_{L(r)}$, a contradiction. Hence such an overlap cannot occur.

no	no	no	no	no	no	*OK*	*OK*	*OK*	*OK*	*OK*	*OK*
1	1	0				0	1	0	0	1	1
1	0	1	1	0	0	0	0	1	0	1	1
0	1	1	0	1	0	0	0	0	1	1	1
			0	0	1	0	0	1	1	0	1

Given that each triple of the remaining rows of C_r rows must be in case P_2 or P_{11}, we must have all triples satisfy only one of the two. ■

Lemma 5.3. *Assume all triples in $L(r)$ are in case P_3. Then the rows of C_r can be ordered so that each triple of rows $a < b < c$ corresponds to $a = i$, $b = j$, and $c = k$ in P_3.*

Proof: In this case there is an ordering of the rows $L(r)$ so that all triples are consistent with the ordering given. We had noted that having P_3 on rows i, j, k in that order correspond to three columns, each on two rows, being absent. If we cannot find a consistent ordering of the rows of $L(r)$, then on some pair of rows we will be missing two columns and this implies that one of the two rows can be deleted while preserving simplicity of $C_r|_{L(r)}$. This contradiction proves the result. ■

In view of Lemma 5.2, we will say $L(r)$ is *type* i if each triple of rows in $L(r)$ is in case P_i for $i = 2, 3$ or 11. Recall we assumed $\|C_r\| \geq 8$. We obtain $M(r)$ from $L(r)$ as follows where the type of $M(r)$ is the type of $L(r)$.

$$M(r) = \begin{cases} L(r) & \text{if } L(r) \text{ is type 2 or 11} \\ L(r)\backslash\{\text{first and last row in ordering}\} & \text{if } L(r) \text{ is type 3} \end{cases} \tag{5.4}$$

Lemma 5.4. *Let $A \in \mathrm{Avoid}(m, \{H_2, H_3, H_5\})$ with (3.1) applied for row r and $M(r)$ from (5.4).*
i) If $M(r)$ is type 2, then $C_r|_{M(r)}$ must consist of $[\mathbf{0}_{|M(r)|}\, I_{|M(r)|}]$ and possibly column $\mathbf{1}_{|M(r)|}$ and no other column. Thus $\|C_r\| - 2 \leq |M(r)| \leq \|C_r\| - 1$. In addition, columns of $A|_{M(r)}$ are from $[\mathbf{0}_{|M(r)|}\, I_{|M(r)|}\, \mathbf{1}_{|M(r)|}]$.
ii) If $M(r)$ is type 11, then $C_r|_{M(r)}$ must consist of $[I^c_{|M(r)|}\, \mathbf{1}_{|M(r)|}]$ and possibly column $\mathbf{0}_{|M(r)|}$ and no other column. Thus $\|C_r\| - 2 \leq |M(r)| \leq \|C_r\| - 1$. In addition columns of $A|_{M(r)}$ are from $[\mathbf{0}_{|M(r)|}\, I^c_{|M(r)|}\, \mathbf{1}_{|M(r)|}]$.

iii) If $M(r)$ is type 3, then $C_r|_{M(r)}$ must consist of $[\mathbf{0}_{|M(r)|}\,\mathbf{0}_{|M(r)|}\,T_{|M(r)|}\,\mathbf{1}_{|M(r)|}]$. Thus $|M(r)| = \|C_r\| - 3$. In addition, columns of $A|_{M(r)}$ are from $[\mathbf{0}_{|M(r)|}\,T_{|M(r)|}]$.

Proof: For $M(r)$ being type 2, we observe that columns of $C_r|_{M(r)}$ must belong to $[\mathbf{0}_{|M(r)|}\,I_{|M(r)|}\mathbf{1}_{|M(r)|}]$. By minimality of $L(r)$ (which is $M(r)$), we cannot delete any rows from $C_r|_{M(r)}$ and preserve simplicity. Thus all columns of $[\mathbf{0}_{|M(r)|}\,I_{|M(r)|}]$ must be present.

A quick count reveals $\|C_r\| - 2 \le |M(r)| \le \|C_r\| - 1$. Similarly for $M(r)$ being type 11, $C_r|_{M(r)}$ must consist of $[I^c_{|M(r)|}\,\mathbf{1}_{|M(r)|}]$ and possibly column $\mathbf{0}_{|M(r)|}$ and no other column. For $M(r)$ being type 3 then, with the row ordering of Lemma 5.3, $C_r|_{L(r)}$ must consist of $[\mathbf{0}_{|L(r)|}\,T_{|L(r)|}]$. Hence $C_r|_{M(r)}$ must consist of $[\mathbf{0}_{|M(r)|}\,\mathbf{0}_{|M(r)|}\,T_{|M(r)|}\,\mathbf{1}_{|M(r)|}]$ and $|M(r)| = \|C_r\| - 3$.

The restricted columns on $C_r|_{M(r)}$ extend to restricted columns on $A|_{M(r)}$ as follows. If $M(r)$ is type 2 then for any $H \subseteq M(r)$ with $|H| = 3$, the 6 forbidden columns on rows $r \cup H$ of Q_2 yield the restrictions P_2 of 3 forbidden columns on rows H of A. Thus the columns of $A|_{M(r)}$ are all contained in $[\mathbf{0}_{|M(r)|}\,I_{|M(r)|}\,\mathbf{1}_{|M(r)|}]$. In a similar way, if $M(r)$ is type 11 then the columns of $A|_{M(r)}$ are all contained in $[\mathbf{0}_{|M(r)|}\,I^c_{|M(r)|}\,\mathbf{1}_{|M(r)|}]$.

If $L(r)$ is type 3 we noted $C_r|_{L(r)}$ is $[\mathbf{0}_{|L(r)|}\,T_{|L(r)|}]$. Indeed, by Lemma 5.3, Q_3 has each triple $i, j, k \in L(r)$ ordered consistent with the ordering of the rows of $L(r)$ yielding T. We deduce the following columns are absent in A on rows $i < j < k$:

$$\begin{matrix} i \\ j \\ k \end{matrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} i \\ j \\ k \end{matrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The following two columns are also forbidden on the 4 rows r, i, j, k of A by Q_3:

$$\alpha = \begin{matrix} r \\ i \\ j \\ k \end{matrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \beta = \begin{matrix} r \\ i \\ j \\ k \end{matrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, using α, under the 0's in row r in $[B_rC_r]|_{L(r)}$ we may only have the columns of $[\mathbf{0}_{|L(r)|}\,T_{|L(r)|}]$ plus one additional column consisting of all 0's except a 1 in the last row of $L(r)$. Similarly using β, under the 1's in row r in $[C_rD_r]|_{L(r)}$ we may only have the columns of $[\mathbf{0}_{|L(r)|}\,T_{|L(r)|}]$ plus one additional column consisting of all 1's except a 0 in the first row of $L(r)$. Thus if $M(r)$ is $L(r)$ with the first and last row deleted then $C_r|_{M(r)} = [\mathbf{0}\,\mathbf{0}\,T\,\mathbf{1}]$ and the columns of $A|_{M(r)}$ are contained in $[\mathbf{0}_{M(r)}\,T_{M(r)}]$. ■

Proof of Lemma 3.1: Let $A \in \text{Avoid}(m, \{H_2, H_3, H_5\})$. Use the decomposition of A given in (3.1). Our procedure is as follows. We use Lemma 5.2 to deduce the possible cases we need to consider. Under the assumption that $\|C_r\| \ge 8$ for all rows r, we will establish by induction an infinite sequence $r_1, r_2, r_3, \ldots$ and associated sets of rows $N(r_1), N(r_2), N(r_3), \ldots$ with $|N(r_i)| \ge 4$ for each i. The sets $N(r)$ differ very little from $L(r)$ and $M(r)$. We are able to show that the sets $N(r_1)\backslash r_2$, $N(r_2)\backslash r_3$, …, $N(r_i)\backslash r_{i+1}$ are all disjoint (see the beginning of Case 1a) and yet $|N(r_j)\backslash r_{j+1}| \ge 3$. This yields a contradiction (there are only m rows!) and so we may conclude that for some r, $\|C_r\| \leq 7$. Hence by our induction we deduce that $\|A\| \leq 7m$.

Assume for all rows r that $\|C_r\| \geq 8$ and hence find the sets $M(r)$ with $|M(r)| \geq 5$ (checking the three cases of Lemma 5.4). Let r_1 be some row of A. We form $M(r_1)$. Note that if $M(r_1)$ was type 3 then we have deleted the first and last rows (in the ordering) from the originally determined $L(r_1)$. We determine the sets $N(r_i)$ from $M(r_i)$ as follows

$$N(r) = \begin{cases} M(r) & \text{if } M(r) \text{ is type 2 or 11} \\ M(r)\backslash \text{ last row in ordering} & \text{if } M(r) \text{ is type 3} \end{cases} \tag{5.5}$$

Our general step commences with $N(r_i)$. We select a row $r_{i+1} \in N(r_i)$, making sure that when $N(r_i)$ is of type 3, we select the first row in the ordering of Lemma 5.3.

Then we obtain $M(r_{i+1})$ applying Lemma 5.1, Lemma 5.2, Lemma 5.3 and Lemma 5.4. Given our assumption that $\|C_r\| \geq 8$ we have $|M(r_{i+1})| \geq 5$. Now by (5.5) we deduce $|N(r_{i+1})| \geq 4$ in all cases. We hope identifying $L(r), M(r), N(r)$ makes the proof clearer.

To show the desired properties of the sets $N(r_i)$, we set up an inductive hypothesis concerning the structure of A. In what follows let Z denote a matrix of 0's (or perhaps a matrix of no columns) and J denote a matrix of 1's (or perhaps a matrix of no columns). The critical inductive structure is the following, for diagrammatic purposes given with a $N(r_p)$ (with $p < i$) type 2 or 3 and $N(r_q)$ (with $q < i$) type 11. The middle columns correspond to the columns of C_{r_i} as shown in (5.6). We have three cases depending on the type of $N(r_i)$. When $N(r_i)$ is type 2 we have $S = [\mathbf{0}\, I]$ or $[\mathbf{0}\, I\, \mathbf{1}]$ and the columns of U_i and V_i are in $[\mathbf{0}\, I\, \mathbf{1}]$. When $N(r_i)$ is type 11 we have $S = [I^c\, \mathbf{1}]$ or $[\mathbf{0}\, I^c\, \mathbf{1}]$ and the columns of U_i, V_i are in $[\mathbf{0}\, I^c\, \mathbf{1}]$. When $N(r_i)$ is type 3 we have $S = [\mathbf{0}\, \mathbf{0}\, T\, \mathbf{1}]$ and the columns of U_i, V_i are in $S = [\mathbf{0}\, T]$.

$$A = \begin{array}{r} r_i \rightarrow \\ \vdots \\ N(r_p)\backslash r_{p+1}\, \{ \\ \vdots \\ N(r_q)\backslash r_{q+1}\, \{ \\ \vdots \\ N(r_i)\, \{ \\ \vdots \end{array} \left[\begin{array}{cc|c|c|cc} 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & 1\cdots 1 & 1\cdots 1 \\ \cline{3-4} \vdots & & & & & \\ Z & W_p^0 & Z & Z & W_p^1 & Z \\ \vdots & & & & & \\ J & W_q^0 & J & J & W_q^1 & J \\ \vdots & & & & & \\ U_i & ZJ & S & S & ZJ & V_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \tag{5.6}$$
$$\qquad\qquad\qquad \underbrace{\quad}_{C_{r_i}} \quad \underbrace{\quad}_{C_{r_i}}$$

We proceed to verify that we have the same inductive structure for r_{i+1}. There will be cases to explore. It is helpful to display representatives of H_2, H_3, H_5 that we will use in our arguments. For $M(r_{i+1})$ type 2 or 11 we will use

$$H_2 = \begin{array}{c} r_{i+1} \\ s \\ i \\ j \end{array} \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right], \quad H_3 = \begin{array}{c} r_{i+1} \\ s \\ i \\ j \end{array} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{5.7}$$

$$H_3 = \begin{array}{c} r_{i+1} \\ t \\ i \\ j \end{array} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right], \quad H_5 = \begin{array}{c} r_{i+1} \\ t \\ i \\ j \end{array} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \tag{5.8}$$

For $M(r_{i+1})$ type 3 we will use

$$H_3 = \begin{array}{c} r_{i+1} \\ s \\ i \\ j \end{array} \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad H_3 = \begin{array}{c} r_{i+1} \\ s \\ i \\ j \end{array} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right] \tag{5.9}$$

$$H_3 = \begin{array}{c} r_{i+1} \\ t \\ i \\ j \end{array} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right], \quad H_3 = \begin{array}{c} r_{i+1} \\ t \\ i \\ j \end{array} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{5.10}$$

Case 1: $N(r_i)$ is type 2.

Begin with inductive structure of (5.6). Given $N(r_i)$ is type 2 we have $S = [\mathbf{0}\, I]$ or $[\mathbf{0}\, I\, \mathbf{1}]$. Choose a row $r_{i+1} \in N(r_i)$. Now consider the decomposition (3.1) applied to A using row $r = r_{i+1}$. Apply Lemma 5.1, Lemma 5.2, Lemma 5.3 and Lemma 5.4 to obtain $M(r_{i+1})$.

Case 1a: $M(r_{i+1})$ is type 2.

The columns of $C_{r_{i+1}}$ must appear once with a 0 in row r_{i+1} and once with a 1 in row r_{i+1}. By Lemma 5.4 we know that columns of $A|_{N(r_i)}$ are contained in $[\mathbf{0}\, I\, \mathbf{1}]$. The only columns of $A|_{N(r_i)}$ which differ only in row r_{i+1} would be the column of 0's and the column of all 0's except a 1 in row r_{i+1}. Thus the repeated columns of $C_{r_{i+1}}$, when restricted to rows $N(r_i)\backslash r_{i+1}$, must be all 0's. By examining (5.6), the only columns of A which on rows $N(r_i)$ that have a single 1 (on row r_{i+1}) on the rows $N(r_i)$ are the columns which are Z in rows $N(r_p)\backslash r_{p+1}$ for those $p < i$ with $N(r_p)$ being type 2 or 3 and J in rows $N(r_q)\backslash r_{q+1}$ for those $q < i$ with $N(r_q)$ being type 11.

We need to show that $N(r_{i+1})$ is disjoint from $N(r_j)\backslash r_{j+1}$ for all $j < i + 1$. All columns in W^0 or W^1 of (5.6) are either all 0's or all 1's on the rows of $N(r_i)$ and so won't give rise to columns of $C_{r_{i+1}}$. We deduce that the columns of $C_{r_{i+1}}$ are all 0's in rows $N(r_p)\backslash r_{p+1}$ for those $p < i$ with $N(r_p)$ being type 2 or 3 and all 1's in rows $N(r_q)\backslash r_{q+1}$ for those $q < i$ with $N(r_q)$ being type 11. Recalling that we form $L(r_{i+1})$ by deleting rows of $C_{r_{i+1}}$ while preserving simplicity, we deduce that $L(r_{i+1})$ (and hence $M(r_{i+1})$ and $N(r_{i+1})$) is disjoint from $N(r_j)\backslash r_{j+1}$ for all $j < i + 1$.

This gives us the structure of $C_{r_{i+1}}$ given below in (5.11) where the two copies of $C_{r_{i+1}}$ occupy the central columns. To complete (5.11) we define W^0 and W^1 (likely different from those in (5.6) in the paragraph above). We choose from the columns of $B_{r_{i+1}}$ and $D_{r_{i+1}}$, all columns which for some $\ell < i$, where $N(r_\ell)$ is type 2 or 3 (and hence rows $N(r_\ell)$ is Z in A), have a 1 in some row of $N(r_\ell)$ or for some $\ell < i$, with $N(r_\ell)$ is type 11 (and hence rows $N(r_\ell)$ is J in A), have a 0 in some row of $N(r_\ell)$. We identify such columns in $B_{r_{i+1}}$ as W^0 and such columns in $D_{r_{i+1}}$ as W^1. Moreover let W_t^0 (respectively W_t^1) denotes the submatrix of W^0 (respectively W^1) in rows $N(r_t)\backslash r_{t+1}$ for $t = 1, \ldots, i$ or in rows $M(r_t)$ for $t = i+1$. All remaining columns of $B_{r_{i+1}}$ and $D_{r_{i+1}}$ are all 0's on rows of each $N(r_\ell)$ where $N(r_\ell)$ is type 2 or 3 and all 1's on rows of each $N(r_\ell)$ where $N(r_\ell)$ is type 11 for $\ell < i$.

$$\begin{array}{r|cccccc|}
r_{i+1} \rightarrow & 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1 & 1\cdots1 \\
\vdots & & & & & & \\
N(r_p)\backslash r_{p+1}\, \{ & Z & W_p^0 & Z & Z & W_p^1 & Z \\
\vdots & & & & & & \\
N(r_q)\backslash r_{q+1}\, \{ & J & W_q^0 & J & J & W_q^1 & J \\
\vdots & & & & & & \\
N(r_i)\backslash r_{i+1} \quad \{ & Z & W_i^0 & Z & Z & W_i^1 & Z \\
M(r_{i+1}) \quad \{ & U_{i+1} & W_{i+1}^0 & \mathbf{0}\, I\, \mathbf{1} & \mathbf{0}\, I\, \mathbf{1} & W_{i+1}^1 & V_{i+1} \\
\vdots & \vdots & & & & &
\end{array} \tag{5.11}$$

By Lemma 5.4 we know that columns of $A|_{N(r_i)}$ are contained in $[\mathbf{0}\, I\, \mathbf{1}]$ and so we deduce that columns of U_{i+1}, V_{i+1} are in $[\mathbf{0}\, I\, \mathbf{1}]$. Our remaining goal is to show that $W_{i+1}^0 = ZJ$ and $W_{i+1}^1 = ZJ$ to complete the induction. We will use the four forbidden matrices of (5.7),(5.8) which have been ordered and labelled to assist the reader in seeing the occurrence of the forbidden objects H_2, H_3, H_5. Assume for some column α of W^0 that α has a 1 in row $s \in N(r_p)\backslash r_{p+1}$ where $N(r_p)$ is type 2 or 3. We will give this first case in greater detail. All columns of $C_{r_{i+1}}$ have 0's in the rows of $N(r_p)$ and in particular in row s. Given that $M(r_{i+1})$ is type 2 or 11 we deduce $C_{r_{i+1}}|_{M(r_{i+1})}$ contains either I or I^c. Thus each pair of rows $i, j \in M(r_{i+1})$ will contain $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ in each copy of $C_{r_{i+1}}$. We find the following entries in A in the rows r, s, i, j where the left column comes from α and the remaining columns are from the two copies of $C_{r_{i+1}}$:

$$\begin{array}{c}
r_{i+1} \\ s \\ i \\ j
\end{array}
\left[\begin{array}{c|cccc}
0 & 0 & 0 & 1 & 1 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
a & 1 & 0 & 1 & 0 \\
b & 0 & 1 & 0 & 1
\end{array}\right].$$

If $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ then we have a representative of H_2 as noted in the left matrix of (5.7). Thus the column α which contains a 1 in some row s of W_p^0 must either be all 0's or all 1's on the rows $M(r_{i+1})$. Assume for some column β of W^0 that β has a 0 in row $t \in N(r_q)\backslash r_{q+1}$ where $N(r_q)$ is type 11. Using the left matrix of (5.8) we may argue as above that column β must either be all 0's or all 1's on the rows $M(r_{i+1})$. Given our choice of W^0, this is enough to show that W^0_{i+1} is ZJ.

Assume for some column α of W^1 that α has a 1 in row $s \in N(r_p)\backslash r_{p+1}$ where $N(r_p)$ is type 2 or 3 and hence we find 0's in $C_{r_{i+1}}$ in row s. Hence by the right matrix in (5.7) we cannot have the matrix $\begin{smallmatrix} i \\ j \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ in α for any choices $i, j \in M(r_{i+1})$. As above, the column α is either all 1's or all 0's on the rows of $M(r_{i+1})$. Similarly, using the right matrix of (5.8) , we can show that for any column β of W^1 that has a 0 in row $t \in N(r_q)\backslash r_{q+1}$ where $N(r_q)$ is type 11 that β cannot have the matrix $\begin{smallmatrix} i \\ j \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ in α for any choices $i, j \in M(r_{i+1})$. Hence β is either all 0's or all 1's on the rows of $M(r_{i+1})$. Thus $W^1_{i+1} = ZJ$ as desired. Setting $N(r_{i+1}) = M(r_{i+1})$ results in the same structure of (5.6) with r_i replaced by r_{i+1} and $S = [\mathbf{0}\, I]$ or $[\mathbf{0}\, I\, \mathbf{1}]$.

Case 1b: $M(r_{i+1})$ is type 11.

We can use the argument of Case 1a if $M(r_{i+1})$ is type 11 since any two rows of I^c contain I_2 allowing us to use the matrices of (5.7),(5.8) as above. We would obtain (5.6) with r_i replaced by r_{i+1}, $N(r_{i+1}) = M(r_{i+1})$ and $S = [I^c\, \mathbf{1}]$ or $[\mathbf{0}\, I^c\, \mathbf{1}]$.

Case 1c: $M(r_{i+1})$ is type 3.

We follow the argument at the beginning of Case 1a) to obtain most of the structure of (5.12). Given that we form $L(r_{i+1})$ by deleting rows of $C_{r_{i+1}}$ while preserving simplicity, we deduce that $L(r_{i+1})$ (and hence $M(r_{i+1})$) is disjoint from $N(r_j)\backslash r_{j+1}$ for all $j < i+1$. We will use (5.9) and (5.10) and, arising from the left matrix of (5.10), we discover a row of $M(r_{i+1})$ that must be deleted.

$$\begin{array}{r|cccccc|}
r_{i+1} \rightarrow & 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1 & 1\cdots1 \\
\vdots & & & & & & \\
N(r_p)\backslash r_{p+1}\,\{ & Z & W^0_p & Z & Z & W^1_p & Z \\
\vdots & & & & & & \\
N(r_q)\backslash r_{q+1}\,\{ & J & W^0_q & J & J & W^1_q & J \\
\vdots & & & & & & \\
N(r_i)\backslash r_{i+1}\quad\{ & Z & W^0_i & Z & Z & W^1_i & Z \\
M(r_{i+1})\quad\{ & U_{i+1} & W^0_{i+1} & \mathbf{0}\,\mathbf{0}\,T\,\mathbf{1} & \mathbf{0}\,\mathbf{0}\,T\,\mathbf{1} & W^1_{i+1} & V_{i+1}
\end{array} \qquad (5.12)$$

Do not be concerned that $C_{r_{i+1}}$ as shown is not simple, as we have deleted two rows from $L(r_{i+1})$ to obtain $M(r_{i+1})$ which are not displayed here. As before, we note that by Lemma 5.4, that the columns of $U_{i+1}, V_{i+1}, W^0_{i+1}, W^1_{i+1}$ are contained in $[\mathbf{0}\, T]$. Our goal to complete the induction is to show $W^0_{i+1} = ZJ$ and $W^1_{i+1} = ZJ$. We use the four forbidden matrices of (5.9),(5.10).

Given that $C_{r_{i+1}}|_{M(r_{i+1})} = [\mathbf{0}\,\mathbf{0}\, T\, \mathbf{1}]$, each pair of rows $i, j \in M(r_{i+1})$ with $i < j$ in the special row ordering of $M(r_{i+1})$ will contain $\left[\begin{smallmatrix} 0\,1\,1 \\ 0\,0\,1 \end{smallmatrix}\right]$ in each copy of $C_{r_{i+1}}$.

If we have a column α of W^1 with a 1 in a row $s \in N(r_j)\backslash r_{j+1}$ where $N(r_j)$ is type 2 or 3 and hence we find 0's in columns of $C_{r_{i+1}}$ in row s. Hence by the right matrix in (5.9), α cannot have the submatrix $\begin{smallmatrix} i \\ j \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for each pair of rows $i, j \in M(r_{i+1})$ with $i < j$. Given that $\alpha|_{M(r_{i+1})}$ is a column in $[\mathbf{0}\, T]$, we deduce that column α is either all

1's or all 0's on the rows of $M(r_{i+1})$. If we have a column β of W^1 with a 0 in a row $t \in N(r_j)\backslash r_{j+1}$ where $N(r_j)$ is type 11, we find 1's in row t of $C_{r_{i+1}}$. Hence by the right matrix in (5.10), β cannot have the submatrix $\begin{smallmatrix} i \\ j \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for each pair of rows $i, j \in M(r_{i+1})$ with $i < j$. As above, the column β is either all 1's or all 0's on the rows of $M(r_{i+1})$. This considers all columns of W^1 and so $W^1_{i+1} = ZJ$.

If we have a column α of W^0 with a 1 in a row $s \in M(r_p)\backslash r_{p+1}$ where $M(r_p)$ is type 2 or 3, we find 0's in row s of $C_{r_{i+1}}$. Hence by the left matrix in (5.9), α cannot have the submatrix $\begin{smallmatrix} i \\ j \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for each pair of rows $i, j \in M(r_{i+1})$ with $i < j$ and so the column α is either all 1's or all 0's on the rows of $M(r_{i+1})$. If we have a column β of W^0 with a 0 in row $t \in N(r_q)$ where $N(r_q)$ is type 11 then we follow a different argument that we explain more carefully. For $i, j \in M(r_{i+1})$ with $i < j$, we find the entries as given below in the rows r_{i+1}, t, i, j in the given column β (the column on the left) and selected columns of $C_{r_{i+1}}$ (on the right).

$$\begin{array}{c} r_{i+1} \\ t \\ i \\ j \end{array} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline a & 0 & 1 & 1 \\ b & 0 & 0 & 0 \end{array}\right]$$

If $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ then this yields H_3 in A as noted in the left matrix in (5.10). Now $\beta|_{M(r_{i+1})}$ is a column in $[\mathbf{0}\, T]$ and yet cannot have the submatrix $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Thus β on the rows of $M(r_{i+1})$ is either all 0's or possibly the column of all 0's except a single 1 in the first row of $M(r_{i+1})$. It is for this case that we need to delete the last row of $M(r_{i+1})$ to obtain $N(r_{i+1})$ (as in (5.5)) so that on the rows $N(r_{i+1})$, the matrix $W^0_{i+1} = ZJ$. We now have obtained (5.6) with r_i replaced by r_{i+1}, and $S = [\mathbf{0}T]$.

Case 2: $N(r_i)$ is type 11.

We use the same argument as Case 1. When $N(r_i)$ is type 11 we would have to replace I by I^c in S in (5.6) and then proceed to $M(r_{i+1})$ of type 2 or 11 (essentially Case 1a or 1b) or $M(r_{i+1})$ of type 3 (essentially Case 1c).

Case 3: $N(r_i)$ is type 3.

Begin with inductive structure of (5.6) where $N(r_i)$ is type 3 and $S = [\mathbf{0}\,\mathbf{0}\,\mathbf{0}\,T\,1\,\mathbf{1}]$. Now choose the *first* row $r_{i+1} \in N(r_i)$ using the ordering on $N(r_i)$. Now consider Standard Induction applied to A using row r_{i+1}. We deduce that in rows $N(r_i)\backslash r_{i+1}$, the repeated columns in $C_{r_{i+1}}$ are Z since for a column to be repeated it extension to row r_{i+1} with both a 0 and a 1 must be present in $C_{r_{i+1}}$. Given that the repeated columns under the 1's in row r_{i+1} must correspond to columns of a single 1 on rows $N(r_i)\backslash r_{i+1}$ and by (5.6) that means we can deduce the structure of the other rows of the columns in $C_{r_{i+1}}$. Note that in what follows I have rearranged the columns of $B_{r_{i+1}}$ and $D_{r_{i+1}}$ so that we have put in the columns of the W_j's all columns which either have a 0 in a row of $N(r_j)$ where $N(r_j)$ is type 2 or 3 (and hence is Z in $C_{r_{i+1}}$), and all columns which have a 1 in a row of $N(r_j)$ where $N(r_j)$ is type 11 (and hence is J in $C_{r_{i+1}}$). This yields (5.11) when $M(r_{i+1})$ is type 2 or (5.12) when $M(r_{i+1})$ is type 3.

If $M(r_{i+1})$ is type 2 we follow the same argument as in Case 1a) to deduce that W^0_{i+1} and W^1_{i+1} have only constant columns. Similarly the case $M(r_{i+1})$ is type 11 can use the argument of Case 1b) by switching I with I^c. In either case we set $N(r_{i+1}) = M(r_{i+1})$. If $M(r_{i+1})$ is type 3, we follow the same argument as in Case 1c) and again may have to delete the first row of $M(r_{i+1})$ to obtain $N(r_{i+1})$ and yields (5.6) with r_i replaces by r_{i+1}. This concludes the induction and so have proven that we can find rows $r_1, r_2, r_3, \ldots$ and disjoint sets $|N(r_i)\backslash r_{i+1}| \geq 3$ yielding a contradiction. As noted this proves the result. ∎

We still have eight 5×6 simple F for which the conjecture predicts they are boundary cases with $\text{forb}(m, F)$ being $O(m^2)$. Given the complicated case analysis of this paper, it seems a daunting prospect to prove such bounds. One positive observation is that Lemma 5.2 may not be necessary. We were only interested in having a large set $L(r)$, say $|L(r)| \geq 8$, for which each triple is in a given case. We could appeal to Ramsey Theory and given a finite number of cases, we can identify a large (!) constant c so that if $\|C_r\| \geq c$ then there are say 8 rows such that every triple is in the same case and in the same row ordering. This would avoid appealing to the particular structures of cases P_2, P_3, P_{11} but is not advantageous for our proof.

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