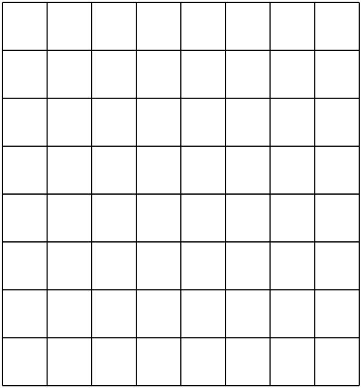
Student Research

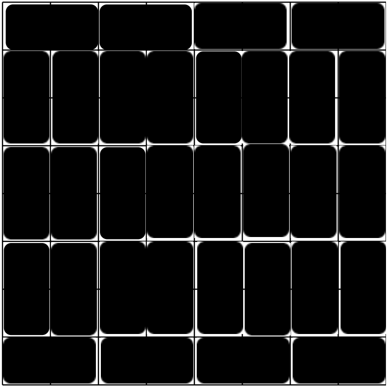
Richard Anstee
UBC, Vancouver

Math Club
Cal State LA, October 24, 2012

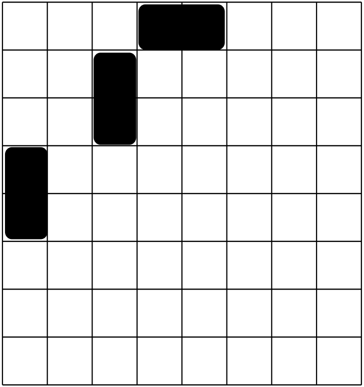
The first set of problems I'd like to mention are really graph theory problems disguised as covering a checkerboard with dominoes. Let me start with the dominoes version



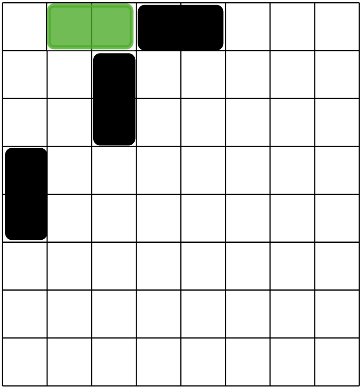
The checkerboard



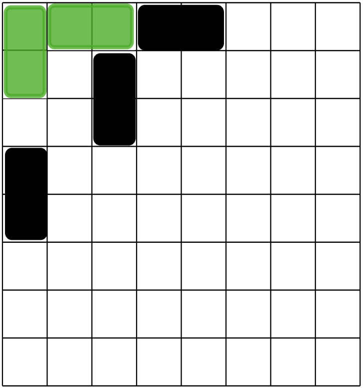
The checkerboard completely covered by dominoes



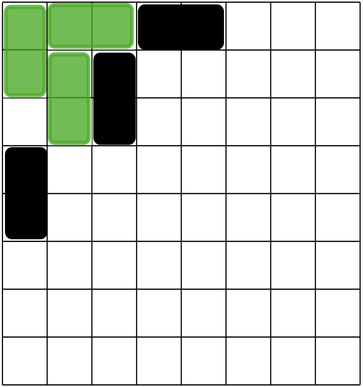
Black dominoes fixed in position. Can you complete?



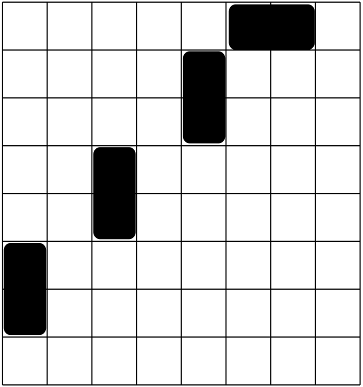
Black dominoes fixed in position. Can you complete?



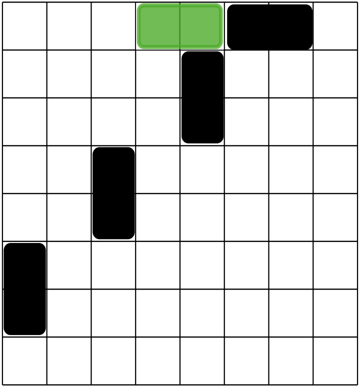
Black dominoes fixed in position. Can you complete?



Black dominoes fixed in position. You can't complete.

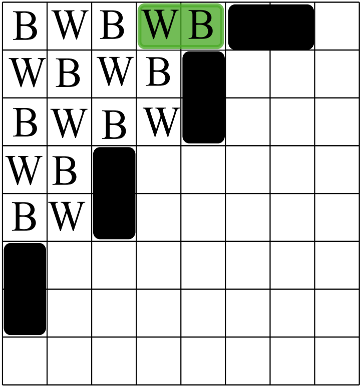


Black dominoes fixed in position. Can you complete?



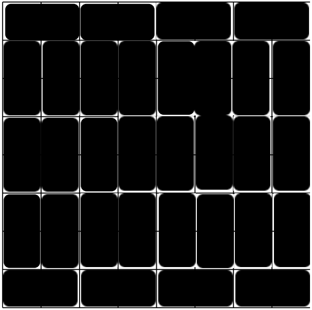
.

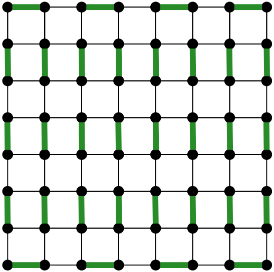
Black dominoes fixed in position. Can you complete?



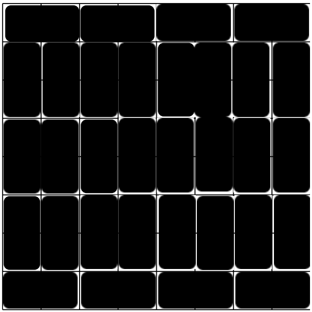
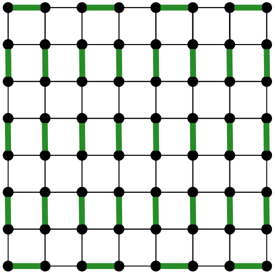
Black dominoes fixed in position. You can't complete.

Covering the checkerboard by dominoes is the same as finding a perfect matching in the associated grid graph.
A perfect matching in a graph is a set M of edges that pair up all the vertices. Necessarily $|M| = |V|/2$.





Covering the checkerboard by dominoes is the same as finding a perfect matching in the associated grid graph.
A perfect matching in a graph is a set M of edges that pair up all the vertices. Necessarily $|M| = |V|/2$.

Theorem (A + Tseng 06) Let m be an even integer. Let S be a selection of edges from the $m \times m$ grid G_m^2. Assume for each pair $e, f \in S$, we have $d(e, f) \geq 3$. Then $G_m^2 \backslash S$ has a perfect matching.

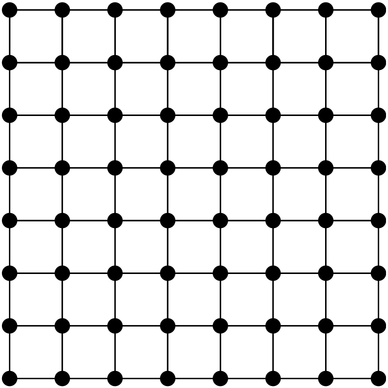
Vertex deletion

Our first example considered choosing some edges and asking whether they extend to a perfect matching. I have also considered what happens if you delete some vertices. Some vertex deletions are clearly not possible. Are there some nice conditions on the vertex deletions so that the remaining graph after the vertex deletions still has a perfect matching?

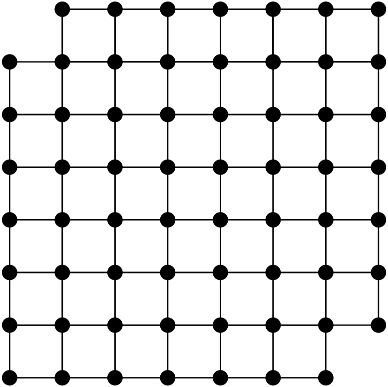
Vertex deletion

Our first example considered choosing some edges and asking whether they extend to a perfect matching. I have also considered what happens if you delete some vertices. Some vertex deletions are clearly not possible. Are there some nice conditions on the vertex deletions so that the remaining graph after the vertex deletions still has a perfect matching?

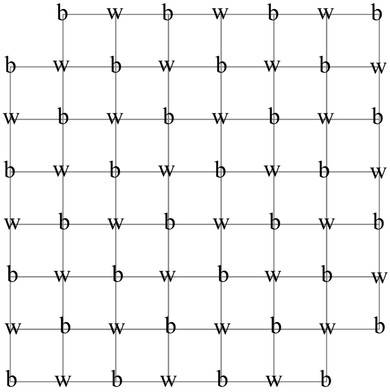
In the checkerboard interpretation we would be deleting some squares from the checkerboard and asking whether the remaining slightly mangled board has a covering by dominoes.



The 8×8 grid.
This graph has many perfect matchings.



The 8×8 grid with two deleted vertices.



The black/white colouring revealed:
No perfect matching in the remaining graph.

Deleting Vertices from Grid

Our grid graph (in 2 or in d dimensions) can have its vertices coloured white W or black B so that every edge in the graph joins a white vertex and a black vertex. Graphs G which can be coloured in this way have $V(G) = W \cup B$ and are called bipartite. Bipartite graphs that have a perfect matching must have $|W| = |B|$.

Thus if we wish to delete black vertices B' and white vertices W' from the grid graph, we must delete an equal number of white and black vertices ($|B'| = |W'|$).

Deleting Vertices from Grid

But also you can't do silly things. Consider a corner of the grid with a white vertex. Then if you delete the two adjacent black vertices then there will be no perfect matching. How do you avoid this problem? Our guess was to impose some distance condition on the deleted blacks (and also on the deleted whites).

Deleting Vertices from Grid

Theorem (Aldred, A., Locke 07 ($d = 2$),
A., Blackman, Yang 10 ($d \geq 3$)).
Let m, d be given with m even and $d \geq 2$. Then there exist constant c_d (depending only on d) for which we set

$$k = c_d m^{1/d} \quad \left(k \text{ is } \Theta(m^{1/d})\right).$$

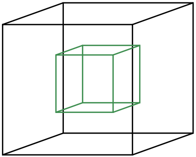
Let G_m^d have bipartition $V(G_m^d) = B \cup W$.
Then for $B' \subset B$ and $W' \subset W$ satisfying
i) $|B'| = |W'|$,
ii) For all $x, y \in B'$, $d(x, y) > k$,
iii) For all $x, y \in W'$, $d(x, y) > k$,
we may conclude that $G_m^d \backslash (B' \cup W')$ has a perfect matching.

Hall's Theorem for G_m^3

The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.

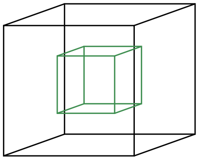
Hall's Theorem for G_m^3

The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.



Hall's Theorem for G_m^3

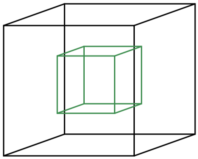
The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.



If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.

Hall's Theorem for G_m^3

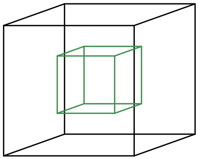
The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.



If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.
If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube $\frac{1}{2}m \times \frac{1}{2}m \times \frac{1}{2}m$.

Hall's Theorem for G_m^3

The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.



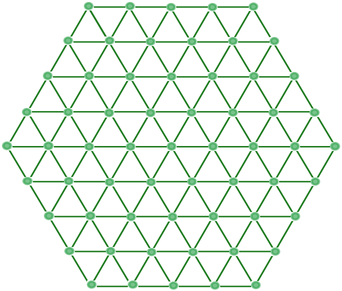
If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.

If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube $\frac{1}{2}m \times \frac{1}{2}m \times \frac{1}{2}m$.

We may choose c small enough so that we cannot find a perfect matching.

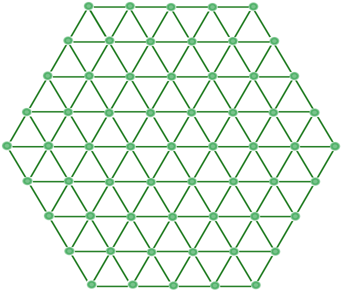
Jonathan Blackman on left

Deleting Vertices from Triangular Grid



A convex portion of the triangular grid

Deleting Vertices from Triangular Grid

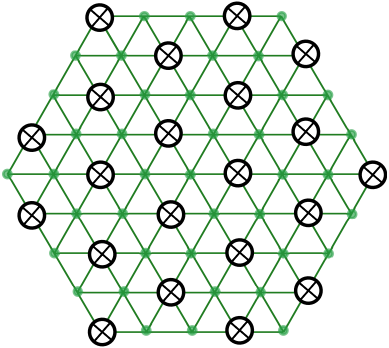


A convex portion of the triangular grid

A *near perfect matching* in a graph is a set of edges such that all but one vertex in the graph is incident with one edge of the matching. Our convex portion of the triangular grid has 61 vertices and many near perfect matchings.

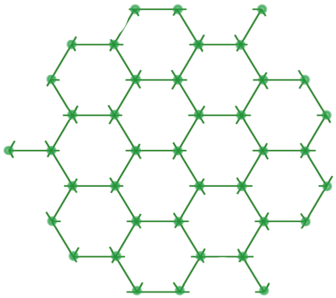
Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



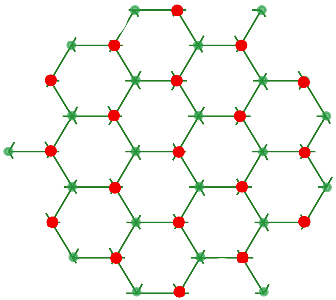
We have deleted 21 vertices from the 61 vertex graph, many at distance 2.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



We have deleted 21 vertices from the 61 vertex graph, many at distance 2.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



We have chosen 19 red vertices S from the remaining 40 vertices and discover that there are 21 other vertices joined only to red vertices and so the 40 vertex graph has no perfect matching.

Extremal Combinatorics

One area I work in is the area of Extremal Set Theory. The typical problem asks how many subsets of $[m] = \{1, 2, \ldots, m\}$ can you choose subject to some property? For example: how many subsets of $[m]$ can you choose such that every pair of subsets has a nonempty intersection?

Extremal Combinatorics

One area I work in is the area of Extremal Set Theory. The typical problem asks how many subsets of $[m] = \{1, 2, \ldots, m\}$ can you choose subject to some property? For example: how many subsets of $[m]$ can you choose such that every pair of subsets has a nonempty intersection?

The answer is $2^{m-1} = \frac{1}{2}2^m$

Extremal Combinatorics

One area I work in is the area of Extremal Set Theory. The typical problem asks how many subsets of $[m] = \{1, 2, \ldots, m\}$ can you choose subject to some property? For example: how many subsets of $[m]$ can you choose such that every pair of subsets has a nonempty intersection?

The answer is $2^{m-1} = \frac{1}{2}2^m$
found by noting that you cannot choose both a set A and its complement $[m]\backslash A$. Easy proof but clever!

A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G. Let Δ denote the triangle on 3 vertices.

The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets.

A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G. Let Δ denote the triangle on 3 vertices.

The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets.

Theorem (Mantel 1907) $ex(m, \Delta) = |E(T(m, 2))| = \lfloor \frac{m^2}{4} \rfloor$

A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G. Let Δ denote the triangle on 3 vertices.

The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets.

Theorem (Mantel 1907) $ex(m, \Delta) = |E(T(m, 2))| = \lfloor \frac{m^2}{4} \rfloor$

Theorem (Turán 41) Let G denote the clique on k vertices (every pair of vertices are joined). Then $ex(m, G) = |E(T(m, k-1))|$.

Let $\chi(G)$ denote the minimum number of colours required to colour the vertices so that no two adjacent vertices have the same colour. Then $\chi(T(m,\ell)) = \ell$. Moreover its is relatively easy to see that $T(m,\ell)$ has the maximum number of edges of all graphs with $\chi = \ell$.

Let $\chi(G)$ denote the minimum number of colours required to colour the vertices so that no two adjacent vertices have the same colour. Then $\chi(T(m,\ell))=\ell$. Moreover its is relatively easy to see that $T(m,\ell)$ has the maximum number of edges of all graphs with $\chi=\ell$.

Also if $\chi(G)=k$, then G is not a subgraph of $T(m,k-1)$, i.e. $ex(m,G)\geq|E(T(m,k-1))|$.

Let $\chi(G)$ denote the minimum number of colours required to colour the vertices so that no two adjacent vertices have the same colour. Then $\chi(T(m,\ell)) = \ell$. Moreover its is relatively easy to see that $T(m,\ell)$ has the maximum number of edges of all graphs with $\chi = \ell$.
Also if $\chi(G) = k$, then G is not a subgraph of $T(m, k-1)$, i.e. $ex(m, G) \geq |E(T(m, k-1))|$.

Theorem (Erdős, Stone, Simonovits 46, 66) Let G be a simple graph. Then

$$\lim_{m\to\infty} \frac{ex(m, G)}{\binom{m}{2}} = 1 - \frac{1}{\chi(G) - 1}.$$

Hypergraphs $\rightarrow$ Simple Matrices

We say $\mathcal{H} = ([m], \mathcal{E})$ is a hypergraph if $\mathcal{E} \subseteq 2^{[m]}$. The subsets in $\mathcal{E}$ are called hyperedges.

Consider a hypergraph $H = ([4], \mathcal{E})$ with vertices $[4] = \{1, 2, 3, 4\}$ and with the following hyperedges :

$$\mathcal{E} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\} \subseteq 2^{[4]}$$

The incidence matrix A of the hyperedges $\mathcal{E} \subseteq 2^{[4]}$ is:

$$A = \left[\begin{array}{cc|c|ccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

Hypergraphs → Simple Matrices

We say $\mathcal{H} = ([m], \mathcal{E})$ is a hypergraph if $\mathcal{E} \subseteq 2^{[m]}$. The subsets in $\mathcal{E}$ are called hyperedges.

Consider a hypergraph $H = ([4], \mathcal{E})$ with vertices $[4] = \{1, 2, 3, 4\}$ and with the following hyperedges :

$$\mathcal{E} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\} \subseteq 2^{[4]}$$

The incidence matrix A of the hyperedges $\mathcal{E} \subseteq 2^{[4]}$ is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{E}|$$

Subhypergraphs $\to$ Configurations

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

We consider the property of forbidding a configuration F in A.

Definition Let

$forb(m,F) = \max\{\|A\| : A \text{ } m\text{-rowed simple, no configuration } F\}$

We consider the property of forbidding a configuration F in A.

Definition Let

$forb(m,F)$= $\max\{\|A\| : A$ m-rowed simple, no configuration $F\}$

$$\text{e.g. } forb(m, \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]) = m+1$$

Some Main Results

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Some Main Results

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\mathit{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

When a matrix A has a copy of K_k on some k-set of rows S, then we say that A shatters S.

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

Let $sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} \boxed{0 \quad 1} & 1 & 1 & 1 & 1 \\ 0 \quad 1 & 0 & 1 & 1 & 0 \\ 0 \quad 0 & 0 & 0 & 1 & 1 \\ 0 \quad 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ \boxed{0 \quad 1} & & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{0 \quad 1} & & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & \boxed{0} & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

Let $sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ \boxed{0 \quad 1 \quad 0 \quad 1} & & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \boxed{0 \quad 1 \quad 1 \quad 0} & & & & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

Let $sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$sh(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}\}$$

So $6 = \|A\| \leq |sh(A)| = 7$

$$\text{Let } sh(A) = \{S \subseteq [m] \;:\; A \text{ shatters } S\}$$

Theorem (Pajor 85) $\quad \|A\| \le |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ A_0 & A_1 \end{array} \right]$$

$$\text{Let } sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$$

Theorem (Pajor 85) $\quad \|A\| \le |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ A_0 & A_1 \end{array}\right]$$

$\|A\| = \|A_0\| + \|A_1\|$.

$$\text{Let } sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$$

Theorem (Pajor 85) $\quad \|A\| \le |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ A_0 & A_1 \end{array}\right]$$

$\|A\| = \|A_0\| + \|A_1\|$.

By induction $\|A_0\| \le |sh(A_0)|$ and $\|A_1\| \le |sh(A_1)|$.

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

Theorem (Pajor 85) $\qquad \|A\| \leq |sh(A)|$.

Proof: Decompose A as follows:

$$A = \begin{bmatrix} 0\,0\cdots 0 & 1\,1\cdots 1 \\ A_0 & A_1 \end{bmatrix}$$

$\|A\| = \|A_0\| + \|A_1\|$.

By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.

$|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|$

Let $sh(A) = \{S \subseteq [m] \; : \; A \text{ shatters } S\}$

Theorem (Pajor 85) $\quad \|A\| \leq |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ A_0 & A_1 \end{array} \right]$$

$\|A\| = \|A_0\| + \|A_1\|$.

By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.

$|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|$

If $S \in sh(A_0) \cap sh(A_1)$, then $1 \cup S \in sh(A)$.

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

Theorem (Pajor 85) $\quad \|A\| \leq |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ A_0 & A_1 \end{array} \right]$$

$\|A\| = \|A_0\| + \|A_1\|$.

By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.

$|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|$

If $S \in sh(A_0) \cap sh(A_1)$, then $1 \cup S \in sh(A)$.

So $(sh(A_0) \cup sh(A_1)) \cup \left(1 + \left(sh(A_0) \cap sh(A_1)\right)\right) \subseteq sh(A)$.

Let $sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$

Theorem (Pajor 85) $\quad \|A\| \leq |sh(A)|$.
Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ A_0 & A_1 \end{array}\right]$$

$\|A\| = \|A_0\| + \|A_1\|$.
By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.
$|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|$
If $S \in sh(A_0) \cap sh(A_1)$, then $1 \cup S \in sh(A)$.
So $(sh(A_0) \cup sh(A_1)) \cup \big(1 + \big(sh(A_0) \cap sh(A_1)\big)\big) \subseteq sh(A)$.
$|sh(A_0)| + |sh(A_1)| \leq |sh(A)|$.

Let $sh(A) = \{S \subseteq [m] \,:\, A \text{ shatters } S\}$

Theorem (Pajor 85) $\quad \|A\| \leq |sh(A)|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ A_0 & A_1 \end{array}\right]$$

$\|A\| = \|A_0\| + \|A_1\|$.
By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.
$|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|$
If $S \in sh(A_0) \cap sh(A_1)$, then $1 \cup S \in sh(A)$.
So $(sh(A_0) \cup sh(A_1)) \cup \left(1 + \left(sh(A_0) \cap sh(A_1)\right)\right) \subseteq sh(A)$.
$|sh(A_0)| + |sh(A_1)| \leq |sh(A)|$.
Hence $\|A\| \leq |sh(A)|$.

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

Proof: Let A have no K_k.

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

Proof: Let A have no K_k.
Then $sh(A)$ can only contain sets of size $k-1$ or smaller.
Then

$$\|A\| \leq |sh(A)| \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Some Main Results

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Some Main Results

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

Some Main Results

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

Some Main Results

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

Problem Given F, can we predict the behaviour of $forb(m, F)$?

Results for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Results for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem (Vapnik and Chervonenkis 71, Perles and Shelah 72, Sauer 72)

$$forb(m, K_4) = \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}.$$

Critical Substructures

We define F' to a critical substructure of F if F' is a configuration in F and

$$forb(m, F') = forb(m, F).$$

Critical Substructures

We define F' to a critical substructure of F if F' is a configuration in F and

$$forb(m, F') = forb(m, F).$$

Note that for F'' which contains F' where F'' is contained in F, we deduce that

$$forb(m, F') = forb(m, F'') = forb(m, F).$$

The critical substructures for K_3 follows from work of A, Karp 10 while the critical substructures for K_4 follows from work of A, Raggi 11. We need some difficult base cases to establish the critical substructures for K_5.

Dr. Miguel Raggi (after Ph.D. defence)

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathit{forb}(m, \mathbf{1}_4) = \mathit{forb}(m, K_4^3) = \mathit{forb}(m, K_4^2) = \mathit{forb}(m, K_4^1)$ $= \mathit{forb}(m, \mathbf{0}_4) = \mathit{forb}(m, 2 \cdot \mathbf{1}_3) = \mathit{forb}(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \left[\begin{array}{|c|ccccccccccccccc} \color{red}{1} & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \color{red}{1} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \color{red}{1} & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \color{red}{1} & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\color{red}{\mathbf{1}_4}$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\textit{forb}(m, \mathbf{1}_4) = \textit{forb}(m, K_4^3) = \textit{forb}(m, K_4^2) = \textit{forb}(m, K_4^1)$ $= \textit{forb}(m, \mathbf{0}_4) = \textit{forb}(m, 2 \cdot \mathbf{1}_3) = \textit{forb}(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \left[\begin{array}{c|cccc|ccccccccccc} 1 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \color{red}1 & \color{red}1 & \color{red}0 & \color{red}1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \color{red}1 & \color{red}0 & \color{red}1 & \color{red}1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & \color{red}0 & \color{red}1 & \color{red}1 & \color{red}1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, $\color{red}{K_4^3}$, K_4^2, K_4^1, $\mathbf{0}_4$, $2\cdot\mathbf{1}_3$, $2\cdot\mathbf{0}_3$.
Note that $forb(m,\mathbf{1}_4) = forb(m,K_4^3) = forb(m,K_4^2) = forb(m,K_4^1)$
$= forb(m,\mathbf{0}_4) = forb(m,2\cdot\mathbf{1}_3) = forb(m,2\cdot\mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \left[\begin{array}{ccccc|cccccc|ccccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$
$= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \left[\begin{array}{ccccccccccc|cccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathit{forb}(m, \mathbf{1}_4) = \mathit{forb}(m, K_4^3) = \mathit{forb}(m, K_4^2) = \mathit{forb}(m, K_4^1)$
$= \mathit{forb}(m, \mathbf{0}_4) = \mathit{forb}(m, 2 \cdot \mathbf{1}_3) = \mathit{forb}(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathit{forb}(m, \mathbf{1}_4) = \mathit{forb}(m, K_4^3) = \mathit{forb}(m, K_4^2) = \mathit{forb}(m, K_4^1)$ $= \mathit{forb}(m, \mathbf{0}_4) = \mathit{forb}(m, 2 \cdot \mathbf{1}_3) = \mathit{forb}(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} \color{red}1 & \color{red}1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \color{red}1 & \color{red}1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \color{red}1 & \color{red}1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $\color{red}{2 \cdot \mathbf{1}_3}$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$
$= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

Critical Substructures for K_4

$$K_4 = \left[\begin{array}{cccccccccccccc|cc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$ $= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

We can extend K_4 and yet have the same bound

Using induction, Connor and I were able to extend the bound of Sauer, Perles and Shelah, Vapnik and Chervonenkis. The base cases of the induction were critical.

Connor Meehan after receiving medal

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have $\mathit{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \mathit{forb}(m, K_4)$.

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have $forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c} 1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0&1\\ 1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0&1\\ 1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&0&0\\ 1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0&0 \end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have
$forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

Are these critical superstructures?

Row and Column order could matter

on the trail with Ronnie Chen

$$\text{Let } F = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We were able to show the following row and column ordered result:

Theorem (A., Chen 11) Let m be given. Let A be an $m \times n$ simple matrix. Assume A has no submatrix F. Then $n \leq \frac{3}{2}m^2 + m + 1$. In addition there is an $m \times (\frac{3}{2}m^2 - 3m)$ simple matrix A with no submatrix F.

$\frac{3}{2}m^2$ is the correct asymptotic bound on n for our F.

Let F be the $2 \times \ell$ matrix $F = \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots \end{bmatrix}$.

We were able to show the following row and column ordered result:

Theorem (A., Estrin 12) Let m be given. Let A be an $m \times n$ simple matrix. Assume A has no submatrix F. Then n is $O(m^2)$ i.e. there exists a constant c_ℓ depending on ℓ so that $n \leq c_\ell m^2$.

$O(m^2)$ is the conjectured asymptotic bound on n for two rowed F.

Ron Estrin

Thank you Daphne for the chance to speak here.