Forbidden Berge hypergraphs

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Abstract

A *simple* matrix is a (0,1)-matrix with no repeated columns. For a (0,1)-matrix F, we say that a (0,1)-matrix A has F as a *Berge hypergraph* if there is a submatrix B of A and some row and column permutation of F, say G, with $G \le B$. Letting $\|A\|$ denote the number of columns in A, we define the extremal function $\mathrm{Bh}(m, F) = \max\{\|A\| : A\ m\text{-rowed simple matrix and no Berge hypergraph } F\}$. We determine the asymptotics of $\mathrm{Bh}(m, F)$ for all 3- and 4-rowed F and most 5-rowed F. For certain F, this becomes the problem of determining the maximum number of copies of K_r in a m-vertex graph that has no $K_{s,t}$ subgraph, a problem studied by Alon and Shinkleman.

Keywords: extremal graphs, Berge hypergraph, forbidden configuration, trace, products

1 Introduction

This paper explores forbidden Berge hypergraphs and their relation to forbidden configurations. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Such a matrix can be viewed as an element-set incidence matrix. Given two (0,1)-matrices F and A, we say A has F as a *Berge hypergraph* and write $F \lll A$ if there is a submatrix B of A and a row and column permutation of F, say G, with $G \le B$. The paper of Gerbner and Palmer [15] introduces this concept to generalize the notions of Berge cycles and Berge paths in hypergraphs. Let F be $k \times \ell$. A Berge hypergraph

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associated with the object F is a hypergraph whose restriction to a set of k elements yields a hypergraph that 'covers' F. Berge hypergraphs are related to the notion of a *pattern* P in a (0,1)-matrix A which has been extensively studied and is quite challenging [14]. We say A has pattern P if there is a submatrix B of A with $P \leq B$. The award winning result of Marcus and Tardos [18] concerns avoiding a pattern corresponding to a permutation matrix. Row and column order matter to patterns.

We use heavily the concept of a *configuration*; see [7]. We say A has a configuration F if there is a submatrix B of A and a row and column permutation of F, say G, with $B = G$. Configurations care about the 0's as well as the 1's in F but do not care about row and column order. In set terminology the notation *trace* can be used.

For a subset of rows S, define $A|_S$ as the submatrix of A consisting of rows S of A. Define $[n] = \{1, 2, \ldots, n\}$ and let $\binom{[n]}{k}$ consist of all k-subsets of $[n]$. If F has k rows and A has m rows and $F \lll A$ then there is a k-subset $S \subseteq [m]$ such that $F \lll A|_S$. For two m-rowed matrices A, B, use $[A \,|\, B]$ to denote the concatenation of A, B yielding a larger m-rowed matrix. Define $t \cdot A = [A\,A \cdots A]$ as the matrix obtained from concatenating t copies of A. Let A^c denote the (0,1)-complement of A. Let $\mathbf{1}_a\mathbf{0}_b$ denote the $(a+b)\times 1$ vector of a 1's on top of b 0's. We use $\mathbf{1}_a$ instead of $\mathbf{1}_a\mathbf{0}_0$. Let K_k^ℓ denote the $k \times \binom{k}{\ell}$ simple matrix of all columns of ℓ 1's on k rows and let $K_k = [K_k^0 K_k^1 K_k^2 \cdots K_k^k]$.

Define $\|A\|$ as the number of columns of A. Define our extremal problem as follows:

$$\mathrm{BAvoid}(m, \mathcal{F}) = \{A \,:\, A \text{ is } m\text{-rowed, simple, } F \not\lll A \text{ for all } F \in F\},$$

$$\mathrm{Bh}(m, \mathcal{F}) = \max_A \{\|A\| \,:\, A \in \mathrm{BAvoid}(m, \mathcal{F})\}.$$

We are mainly interested in $\mathcal{F}$ consisting of a single forbidden Berge hypergraph F. When $|\mathcal{F}| = 1$ and $\mathcal{F} = \{F\}$, we write $\mathrm{BAvoid}(m, F)$ and $\mathrm{Bh}(m, F)$.

The main goal of this paper is to explore the asymptotic growth rate of $\mathrm{Bh}(m, F)$ for a given $k \times \ell$ (0,1)-matrix F. Theorem 3.1 handles $k = 3$, Theorem 4.4 handles $k = 4$ and Theorem 5.1 handles $k = 5$ (modulo Conjecture 7.1). The results apply some of the proof techniques (and results) for Forbidden configurations [7]. We have some interesting connections with $\mathrm{ex}(m, K_{s,t})$ (the maximum number of edges in a graph on m vertices with no complete bipartite graph $K_{s,t}$ as a subgraph) and $\mathrm{ex}(m, K_n, K_{s,t})$ [6] (the maximum number of complete subgraphs K_n in a graph on m vertices with no complete bipartite graph $K_{s,t}$ as a subgraph). Section 6 has some applications of this graph theory such as Theorem 6.1 and Theorem 6.3. Note that K_k has two meanings in this paper that are hopefully clear by context namely either the complete graph on k vertices or as the matrix $[K_k^0 K_k^1 K_k^2 \cdots K_k^k]$. We also obtain in Theorem 6.5, that if F is the vertex-edge incidence matrix of a tree T, then $\mathrm{Bh}(m, F)$ is $\Theta(m)$ analogous to $\mathrm{ex}(m, T)$.

We first make some easy observations.

Remark 1.1 *Let F, F' be two $k \times \ell$ (0,1)-matrices satisfying $F \lll F'$. Then* $\mathrm{Bh}(m, F) \leq \mathrm{Bh}(m, F')$.

The related extremal problem for forbidden configurations is as follows:

$$\text{Avoid}(m, \mathcal{F}) = \{A : A \text{ is } m\text{-rowed, simple}, F \nprec A \text{ for all } F \in F\},$$

$$\text{forb}(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}.$$

When $|\mathcal{F}| = 1$ and $\mathcal{F} = \{F\}$, we write $\text{Avoid}(m, F)$ and $\text{forb}(m, F)$. There are striking differences between $\text{Bh}(m, F)$ and $\text{forb}(m, F)$ such as Theorem 6.5 for Berge hypergraphs and Theorem 6.9 for forbidden configurations. Note that the two notions of Berge hypergraphs and configurations coincide when F has no 0's.

Remark 1.2 *Let F be a (0,1)-matrix. Then* $\text{Bh}(m, F) \le \text{forb}(m, F)$. *If F is a matrix of 1's then* $\text{Bh}(m, F) = \text{forb}(m, F)$.

Note that any forbidden Berge hypergraph F can be given as a family $\mathcal{B}(F)$ of forbidden configurations by replacing the 0's of F by 1's in all possible ways. Define

$$\mathcal{B}(F) = \{B \text{ is a (0,1)-matrix} : F \le B\}. \tag{1}$$

Remark 1.3 $\text{Bh}(m, F) = \text{forb}(m, \mathcal{B}(F))$.

Isomorphism can reduce the required set of matrices to consider, for example $\mathcal{B}(I_2)$ which has 4 matrices satisfies:

$$\text{BAvoid}(m, \mathcal{B}(I_2)) = \text{BAvoid}(m, \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}).$$

A product construction is helpful. Let A, B be $m_1 \times n_1$ and $m_2 \times n_2$ simple matrices respectively. We define $A \times B$ as the $(m_1 + m_2) \times n_1 n_2$ matrix whose columns are obtained by placing a column of A on top of a column of B in all $n_1 n_2$ possible ways. This extends readily to p-fold products. Let $I_t = K_t^1$ denote the $t \times t$ identity matrix. In what follows you may assume p divides m since we are only concerned with asymptotic growth with respect to m.

$$\text{The } p\text{-fold product } \overbrace{I_{m/p} \times I_{m/p} \times \cdots \times I_{m/p}}^{p},$$

is an $m \times m^p/p^p$ simple matrix. This corresponds to the vertex-edge incidence matrix of the complete p-partite hypergraph with parts $V_1, V_2, \ldots, V_p$ each of size m/p so that $\{v_1, v_2, \ldots, v_p\}$ is an edge if and only if $v_i \in V_i$ for $i = 1, 2, \ldots, p$. These products sometimes yield the asymptotically best (in growth rate) constructions avoiding F as a Berge hypergraph.

Remark 1.4 *Let F be a given $k \times \ell$ (0,1)-matrix so that $F \not\leq I_{m/p} \times I_{m/p} \times \cdots \times I_{m/p}$ (a p-fold product). Then* $\text{Bh}(m, F)$ *is* $\Omega(m^p)$.

Sometimes the product may contain the best construction using the following idea from [5], that when given two matrices F, P where P is m-rowed then

$$f(F,P) = \max_A\{\|A\| \mid A \text{ is } m\text{−}rowed, A \prec P \text{ and } F \not\prec A\}.$$

Thus Theorem 4.3 yields $\mathrm{Bh}(m, I_2 \times I_2)$ is $\Theta(f(I_2 \times I_2, I_{m/2} \times I_{m/2}))$ but Lemma 6.2 indicates that things must be more complicated for $F = I_s \times I_t$ for general s, t.

A shifting process works nicely here. Let $T_i(A)$ denote the matrix obtained from A by attempting to replace 1's in row i by 0's . Do not replace a 1 by a 0 in row i and column j if the resulting column is already present in A otherwise do replace the 1 by a 0. Then $\|T_i(A)\| = \|A\|$ and, if A is simple, then $T_i(A)$ is simple.

Lemma 1.5 *Given $A \in \mathrm{BAvoid}(m, F)$, there exists a matrix $T(A) \in \mathrm{BAvoid}(m, F)$ with $\|A\| = \|T(A)\|$ and $T_i(T(A)) = T(A)$ for $i = 1, 2, \ldots, m$.*

Proof: It is automatic that $\|A\| = \|T_i(A)\|$. We note that $F \not\prec A$ implies $F \not\prec T_i(A)$. Replace A by $T_i(A)$ and repeat. Let $T^*(A) = T_m(T_{m-1}(\cdots T_1(A)\cdots))$. Either $T^*(T^*(A))$ contains fewer 1's than $T^*(a)$ or we have $T_i(T^*(A)) = T^*(A)$ for $i = 1, 2, \ldots, m$. In the former case replace A by $T^*(A)$ and repeat. In the latter case let $T(A) = T^*(A)$. Since the number of 1's in A is finite, then the process will terminate with our desired matrix $T(A)$. ∎

Typically $T(A)$ is referred to as a *downset* since when the columns of $T(A)$ are interpreted as a set system $\mathcal{T}$ then if $B \in \mathcal{T}$ and $C \subset B$ then $C \in \mathcal{T}$. Note that if $T(A)$ has a column of sum k with 1's on rows S, then $K_k \prec T(A)|_S$ and moreover the copy of K_k on rows S can be chosen with 0's on all other rows. An easy consequence is that for $A \in \mathrm{BAvoid}(m, F)$ where F is k-rowed and simple then we may assume A has no columns of sum k.

2 General results

This section provides a number of results about Berge hypergraphs that are used in the paper. The following results from forbidden configurations were useful.

Theorem 2.1 *[2] Let k, t be given with $t \geq 2$. Then $\mathrm{forb}(m, t \cdot \mathbf{1}_k) = \mathrm{forb}(m, t \cdot K_k)$ and is $\Theta(m^k)$.*

Theorem 2.2 *[6] Let k, t be given. Then $\mathrm{forb}(m, [\mathbf{1}_k \mid t \cdot K_k^{k-1}])$ is $\Theta(m^{k-1})$.*

Theorem 2.3 *[3] Let F be a k-rowed simple matrix. Assume there is some pair of rows i, j so than no column of F contains 0's on rows i, j, there is some pair of rows i, j so than no column of F contains 1's on rows i, j and there is some pair of rows i, j so than no column of F contains I_2 on rows i, j. Then $\mathrm{forb}(m, F)$ is $O(m^{k-2})$.*

Definition 2.4 *Let F be a k-rowed (0,1)-matrix. Define $G(F)$ as the graph on k vertices such that we join vertices i and j by an edge if and only if there is a column in F with 1's in rows i and j. Let $\omega(G(F))$ denote the size of the largest clique in $G(F)$ and let $\chi(G(F))$ denote the chromatic number of $G(F)$. Let $\alpha(G(F))$ denote the size of the largest independent set in $G(F)$.*

Lemma 2.5 *Let F be given. Then* $\mathrm{Bh}(m, F)$ *is* $\Omega(m^{\chi(G(F))-1})$ *and hence* $\Omega(m^{\omega(G(F))-1})$.

Proof: Let $p = m/(\chi(G(F)) - 1)$. Let

$$A = \overbrace{I_p \times I_p \times \cdots \times I_p}^{\chi(G(F))-1}.$$

Assume $F \prec\!\!\prec A$ and the rows of the $\chi(G(F)) - 1$ fold product containing F are S then we obtain $\chi(G(F)) - 1$ disjoint sets $S_1, S_2, \ldots, S_{X(F)-1}$ with $S_i = S \cap \{(i-1)p+1, (i-1)p+2, \ldots, ip\}$ and $A|_{S_i} \prec\!\!\prec I_{|S_i|}$. This contradicts the definition of $\chi(G(F))$ and so $F \not\prec\!\!\prec A$. Thus $\mathrm{Bh}(m, F)$ is $\Omega(m^{\chi(G(F))-1})$. Note that $\chi(G) \geq \omega(G)$. ■

Lemma 2.6 *If* $2 \cdot \mathbf{1}_t \prec\!\!\prec F$ *then* $\mathrm{Bh}(m, F)$ *is* $\Omega(m^t)$

Proof: F is not a Berge hypergraph of the t-fold product $I_{m/t} \times I_{m/t} \times \cdots \times I_{m/t}$. ■

Theorem 2.7 *Let k be given and assume $m \geq k - 1$. Then* $\mathrm{Bh}(m, I_k) = 2^{k-1}$.

Proof: The construction consisting of K_{k-1} with $m - k + 1$ rows of 0's added yields $\mathrm{Bh}(m, I_k) \geq 2^{k-1}$.

We use induction on k. The largest m-rowed matrix which avoids $I_1 = [1]$ as a Berge hypergraph is $[\mathbf{0}_m]$. This proves the base case $k = 1$ and the following is the inductive step.

Let $A \in \mathrm{BAvoid}(m, I_k)$. Let B be obtained from A by removing any rows of 0's so that B is simple and every row of B contains a 1. If B has $k - 1$ rows then $\|A\| = \|B\| \leq 2^{k-1}$ which is our bound. Assume B has at least k rows. Either $\|B\| \leq 2^{k-1}$ in which case we are done or $\|B\| > 2^{k-1} > 2^{k-2}$ and so by induction, B must contain I_{k-1} as a Berge hypergraph. Permute B to form the block matrix

$$B = \left[\begin{array}{c|c} C & D \\ \hline E & G \end{array}\right]$$

where C is $(k-1) \times (k-1)$ with $I_{k-1} \prec\!\!\prec C$. Then G must be the matrix of 0's or else $I_k \prec\!\!\prec B$. Thus D is simple. Since all rows of B contain a 1, then E must have a 1. If E contains a 1 then $I_{k-1} \not\prec\!\!\prec D$ and so $\|D\| \leq 2^{k-2}$. This gives $\|B\| = \|C\| + \|D\| = k - 1 + 2^{k-2} \leq 2^{k-1}$. Thus $\|A\| = \|B\| \leq 2^{k-1}$. ■

Theorem 2.7 establishes a constant bound for the Berge hypergraph I_k. The existence of a constant bound follows from a result of Balogh and Bollobás [8]. Let $I_k^c = K_k^{k-1}$ denote the $k \times k$ (0,1)-complement of I_k and let T_k denote the $k \times k$ upper triangular (0,1)-matrix with a 1 in row i and column j if and only if $i \leq j$.

Theorem 2.8 *[8] Let k be given. Then there is a constant c_k so that* $\mathrm{forb}(m, \{I_k, I_k^c, T_k\}) = c_k$.

A corollary of Koch and the first author [4] gives one way to apply this result.

Theorem 2.9 *[4] Let $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ be given. There are two possibilities. Either* $\mathrm{forb}(m, \mathcal{F})$ *is $\Omega(m)$ or there exist ℓ, i, j, k with $F_i \prec I_\ell$, with $F_j \prec I_\ell^c$ and with $F_k \prec T_\ell$ in which case there is a constant c with* $\mathrm{forb}(m, \mathcal{F}) = c$.

We apply this result to a $k \times \ell$ forbidden Berge hypergraph F using the family $\mathcal{B}(F)$ from (1) which contains the $k \times \ell$ matrix of 1's. Noting that $I_{k+\ell+1}^c$ contains a $k \times \ell$ block of 1's and $T_{k+\ell}$ contains a $k \times \ell$ block of 1's we obtain the following.

Corollary 2.10 *Let F be a $k \times \ell$ (0,1)-matrix. Then either* $\mathrm{Bh}(m, F)$ *is $\Omega(m)$ or $F \ll I_{k+\ell}$ in which case* $\mathrm{Bh}(m, F)$ *is $O(1)$.*

The following Lemma (from 'standard induction' in [7]) was useful for forbidden configurations.

Lemma 2.11 *Let F be a $k \times \ell$ (0,1)-matrix and let F' be a $(k-1) \times \ell$ submatrix of F. Then* $\mathrm{Bh}(m, F) = O(m \cdot \mathrm{Bh}(m, F'))$.

Proof: Let $A \in \mathrm{BAvoid}(m, F)$. If we delete row 1 of A, then the resulting matrix may have columns that appear twice. We may permute the columns of A so that

$$A = \left[\begin{array}{cc} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B \quad C & C \quad D \end{array} \right],$$

where $[B\,C\,D]$ and C are simple $(m-1)$-rowed matrices. We have $[B\,C\,D] \in \mathrm{BAvoid}(m-1, F)$ and $C \in \mathrm{BAvoid}(m-1, F')$ (if $F' \ll C$ then $F \ll A$). Then

$$\|A\| = \|[BCD]\| + \|C\| \leq \mathrm{Bh}(m-1, F) + \mathrm{Bh}(m-1, F'),$$

which yields the desired bound, by induction on m. ■

Lemma 2.12 *Let A be a k-rowed (0,1)-matrix, not necessarily simple, with all row sums at least kt. Then $t \cdot I_k \ll A$.*

Proof: We use induction on k where the case $k = 1$ and $I_1 = [1]$ is easy. Choose t columns from A containing a 1 in row 1 and remove them and row 1 resulting in a matrix A'. The row sums of A' will be at least $(k-1)t$ and so we may apply induction. Thus $(t-1) \cdot I_k \ll A'$ and so we obtain $t \cdot I_k \ll A$. ■

An interesting corollary is that if we have an m-rowed matrix A with all rows sums at least kt then $t \cdot I_k \ll A|_S$ for all $S \in \binom{[m]}{k}$.

Lemma 2.13 *Let A be a given m-rowed matrix and let $\mathcal{S}$ be a family of subsets of $[m]$ with the property that $|S| \le k$ for all $S \in \mathcal{S}$. Let c be given. Then by deleting at most $c\left(\binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{1}\right)$ columns from A we can obtain a matrix A' so that for each $S \in \mathcal{S}$, $A'|_S$ either has more than c columns with 1's on all the rows of S or has no columns with 1's on all the rows of S.*

Proof: For each subset of $S \in \mathcal{S}$, if the number of columns of $A|_S$ with 1's on the rows of S is at most c, then delete all such columns. Repeat. The number of deleted columns is at most $\sum_{S\in\mathcal{S}} c \le c\left(\binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{1}\right)$. ■

Lemma 2.14 *(Reduction Lemma) Let $F = [G \,|\, t\cdot[H\, K]]$. Assume H, K are simple and have column sums at most k. Also assume for each column α of K, there is a column γ of $[G\, H]$ with $\alpha \le \gamma$. Then there is a constant c so that* $\mathrm{Bh}(m, F) \le \mathrm{Bh}(m, [G\, H]) + cm^k$.

Proof: We let $A \in \mathrm{BAvoid}(m, [G \,|\, t\cdot[H\, K]])$ and $c = \|G\| + t\|H\| + t\|K\|$. Applying Lemma 1.5, assume $T_i(A) = A$ for all i and so, when columns are viewed as sets, the columns form a downset. Form $\mathcal{S}$ as the union of all sets $S \subseteq [m]$ so that $[H\, K]$ has a column with 1's on the rows S. Then, applying Lemma 2.13, delete at most cm^k columns to obtain a matrix A'. Now if $[G\, H] \lll A'$ on rows S, then each column contributing to H will appear c times in $A'|_S$.

Moreover each column γ of G will appear at least c times in $A'|_S$ and so if α is a column of K and γ is a column of G with $\alpha \le \gamma$, then we have $t\cdot\alpha \lll t\cdot\gamma$. Hence $[G \,|\, t\cdot[H\, K]] \lll A|_S$, a contradiction. The choice of c above is required, for example, when the columns contributing to $[G\, H]$ all have $A|_S = \mathbf{1}$. ■

The following are two important applications. We use the notation $K_p\backslash\mathbf{1}_p$ to denote the matrix obtained from K_p by deleting the column of p 1's.

Theorem 2.15 *Let $H(p,k,t) = [1_p \times I_{k-p} \,|\, t\cdot[\,\mathbf{1}_p \times \mathbf{0}_{k-p} \,|\, (K_p\backslash\mathbf{1}_p) \times [\mathbf{0}_{k-p}\; I_{k-p}]]$, i.e.*

$$H(p,k,t) = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{array}\quad t\cdot\left[\begin{array}{cc} 1 & \\ \vdots & K_p\backslash\mathbf{1}_p \\ 1 & \times \\ 0 & \\ 0 & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \vdots & \\ 0 & \end{array}\right]\right]. \tag{2}$$

Then $\mathrm{Bh}(m, H(p,k,t))$ *is* $\Theta(m^p)$. *Moreover if we add to $H(p,k,t)$ any column not already present t times in $H(p,k,t)$ to obtain F', then* $\mathrm{Bh}(m, F')$ *is* $\Omega(m^{p+1})$.

Proof: Let $F = H(p,k,t)$. Given that F has a column of $p+1$ 1's then $\omega(G(F)) \ge p+1$ and so Lemma 2.5 yields $\mathrm{Bh}(m, F)$ is $\Omega(m^p)$.

To apply Reduction Lemma 2.14, set $F = [G \,|\, t \cdot [H\, K]]$ with G to be the first $k-p$ columns of F and with K to be the remaining $1 + (2^p - 1) \times (k-p)$ columns of F when $t=1$ and with H absent. Now $\mathrm{Bh}(m, F) \le \mathrm{Bh}(m, G) + cm^p$ for $c = \|G\| + t\|K\|$. Applying Lemma 2.11 repeatedly (in essence deleting the first p rows of G) we obtain $\mathrm{Bh}(m, G) = O(m^k \mathrm{Bh}(m, I_{k-p}))$ and so with Lemma 2.7 this yields $\mathrm{Bh}(m, G)$ is $O(m^p)$. Then $\mathrm{Bh}(m, H(p, k, t))$ is $\Theta(m^p)$.

The remaining remarks concerning adding a column to $H(p, k, t)$ are covered in Lemma 2.17. ■

Note that $\mathrm{Bh}(m, H(k-1, k, t))$ follows from Theorem 2.2. There is a more general form of $H(p, k, t)$ as follows.

Definition 2.16 *Let A be a given (0,1)-matrix. Let $\mathcal{S}(A)$ denote the matrix of all columns α so that there exists a column γ of A with $\alpha \le \gamma$ and $\alpha \ne \gamma$.* ■

$$\text{Let } H((a_1, a_2, \ldots, a_s), t) = [I_{a_1} \times I_{a_2} \times \cdots \times I_{a_s} \,|\, t \cdot \mathcal{S}([I_{a_1} \times I_{a_2} \times \cdots \times I_{a_s}])]. \tag{3}$$

Then $H(p, k, t)$ is $H((a_1, a_2, \ldots, a_s), t)$ where $s = p+1$ and $a_1 = a_2 = \cdots = a_p = 1$ and $a_{p+1} = k - p$. The upper bounds of Theorem 2.15 do not generalize but the second part of the proof continues to hold.

Lemma 2.17 *Let $H((a_1, a_2, \ldots, a_s), t)$ be defined as in (3). Then $\mathrm{Bh}(m, H((a_1, a_2, \ldots, a_s), t))$ is $\Omega(m^{s-1})$. Moreover if we add to $H((a_1, a_2, \ldots, a_s), t)$ any column α not already present t times in $H((a_1, a_2, \ldots, a_s), t)$ then $\mathrm{Bh}(m, [H((a_1, a_2, \ldots, a_s), t) \,|\, \alpha])$ is $\Omega(m^s)$.*

Proof: The lower bound for $\mathrm{Bh}(m, H((a_1, a_2, \ldots, a_s), t))$ follows from $(s-1)$-fold product $I_{m/(s-1)} \times I_{m/(s-1)} \times \cdots \times I_{m/(s-1)}$ since $H((a_1, a_2, \ldots, a_s), t))$ has columns of sum s.

There are two choices for α. First we can choose α to be a column in $I_{a_1} \times I_{a_2} \times \cdots \times I_{a_s}$ and so α has s 1's. Then $2 \cdot \mathbf{1}_s \prec\!\!\prec [\alpha\alpha]$ so that $\mathrm{Bh}(m, [\alpha\alpha])$ is $\Theta(m^s)$ by Theorem 2.1.

Second choose α to be a column not already present in $H((a_1, a_2, \ldots, a_s), t)$. Let $G = G(H((a_1, a_2, \ldots, a_s), t))$ be the graph defined in Definition 2.4 on $a_1 + a_2 + \cdots + s_s$ vertices corresponding to rows of $H((a_1, a_2, \ldots, a_s), t)$. Our choice of α has a pair of rows h, ℓ so that α has 1's in both rows h and ℓ and the edge h, ℓ is not in G. We deduce that $[H((a_1, a_2, \ldots, a_s), t) \,|\, \alpha]$ has $s+1$ rows S such that for every pair $i, j \in S$, there is a column with 1's in both rows i and j, i.e. G has a clique of size $s+1$. Thus by Lemma 2.5, $\mathrm{Bh}(m, [H((a_1, a_2, \ldots, a_s), t) \,|\, \alpha])$ is $\Omega(m^s)$. ■

Thus all but the upper bounds for Theorem 2.15 follow from Lemma 2.17. The following application requires Conjecture 7.1 to be true. Note that $\mathbf{1}_1 \times C_4$ is $I_1 \times I_2 \times I_2$.

Theorem 2.18 *Assume* $\mathrm{Bh}(m, \mathbf{1}_1 \times C_4)$ *is* $\Theta(m^2)$. *Then* $\mathrm{Bh}(m, H((1,2,2),t))$ *is* $\Theta(m^2)$. *Moreover if we add to* $H((1,2,2),t)$ *any column* α *not already present* t *times in* $H((1,2,2),t)$ *to obtain* $[H((1,2,2),t)\,|\,\alpha]$, *then* $\mathrm{Bh}(m, [H((1,2,2),t)\,|\,\alpha])$ *is* $\Omega(m^3)$.

Proof: Take $G = \mathbf{1}_1 \times I_2 \times I_2 = \mathbf{1}_1 \times C_4$ and take K to be the remainder of the columns of $H((1,2,2),1)$ and then apply Reduction Lemma 2.14 and the hypothesis that $\mathrm{Bh}(m, \mathbf{1}_1 \times C_4)$ is $\Theta(m^2)$ to obtain the upper bound.

The rest follows from Lemma 2.17. $\blacksquare$

The following monotonicty result seems obvious but note that monotonicity is only conjectured to be true for forbidden configurations.

Lemma 2.19 *Assume* F *is a* $k \times \ell$ *matrix and assume* $m \geq k$, *Then* $\mathrm{Bh}(m, F) \geq \mathrm{Bh}(m-1, F)$.

Proof: Let F' be the matrix obtained from F by deleting rows of 0's, if any. Then for $m \geq k$, $A \in \mathrm{BAvoid}(m, F)$ if and only if $A \in \mathrm{BAvoid}(m, F')$. Now assume $A \in \mathrm{BAvoid}(m-1, F')$ with $m \geq k$. Then form A' from A by adding a single row or 0's. Then $A' \in \mathrm{BAvoid}(m, F')$ with $\|A\| = \|A'\|$. $\blacksquare$

The following allows F to have rows of 0's contrasting with Reduction Lemma 2.14.

Lemma 2.20 *Let* F *be a* $k \times \ell$ *matrix. Then* $\mathrm{Bh}(m, [F\,|\,t \cdot I_k]) \leq \mathrm{Bh}(m, F) + (tk + \ell)m$.

Proof: Let $A \in \mathrm{BAvoid}(m, [F\,|\,t \cdot I_k]$. For any row in A of row sum r we may remove that row and the r columns containing a 1 on that row and the remaining $(m-1)$-rowed matrix is simple. In this way remove all rows with row sum at most $tk + l$ and call the remaining simple matrix B and assume it has m' rows. Then $\|A\| \leq \|B\| + (tk + \ell)(m - m')$. Suppose B contains F on some k-rows $S \subseteq \binom{[m']}{k}$. Remove the columns containing F from B to obtain B' and now the rows of B' have row sum $\geq tk$. By Lemma 2.12, $t \cdot I_k$ is contained in $B'|_S$. Consequently $[F\,|\,t \cdot I_k]$ is contained in B. This is a contradiction so we conclude that $B \in \mathrm{BAvoid}(m', F)$. Hence $\|B\| \leq \mathrm{Bh}(m', F) \leq \mathrm{Bh}(m, F)$ (by Lemma 2.19). We also know that $\|B\| \geq \|A\| - (tk + \ell)m$ and so $\|A\| \leq \mathrm{Bh}(m, F) + (tk + \ell)m$ for all A. $\blacksquare$

Lemma 2.21 *Let* F *be a given* k*-rowed (0,1)-matrix. Let* F' *denote the matrix obtained from* F *by adding a row of 0's. Then* $\mathrm{Bh}(m, F') = \mathrm{Bh}(m, F)$ *for* $m > k$. *Also* $\mathrm{Bh}(m, [\mathbf{0}_k\, F]) = \max\{\|F\|, \mathrm{Bh}(m, F))\}$.

Proof: Let A be a simple m-rowed matrix with $\|A\| > \mathrm{Bh}(m, F)$. Then $F \lll A$. Now as long as $m \geq k+1$ we have that $F' \lll A$. Similarly if $\|A\| > \|F\|$, then $[\mathbf{0}_k\, F] \lll A$. $\blacksquare$

A more general result would be the following.

Theorem 2.22 *Let F_1, F_2 be given. For F as below,* $\mathrm{Bh}(m,F)$ *is* $O(\|F_1\| + \|F_2\| + \max\{\mathrm{Bh}(m,F_1), \mathrm{Bh}(m,F_2)\})$.

$$F = \left[\begin{array}{c|c} F_1 & 0 \\ \hline 0 & F_2 \end{array}\right].$$

Proof: Assume F_1 is k-rowed. Let $A \in \mathrm{BAvoid}(m,F)$. If $\|A\| > \mathrm{Bh}(m,F_1)$, then $F_1 \prec A$. Assume F_1 appears in the first k rows so that

$$A = \left[\begin{array}{c|c} F_1 & * \\ \hline * & B \end{array}\right].$$

If $F_2 \prec B$ then $F \prec A$ and so we may assume $F_2 \not\prec B$. Now the multiplicity of any column of B is at most 2^k. Thus $\|B\| \le 2^k \mathrm{Bh}(m,F_2)$ and so $\|A\| \le \|F_1\| + 2^k \mathrm{Bh}(m-k,F_2) \le \|F_1\| + 2^k \mathrm{Bh}(m,F_2)$ by Lemma 2.19. Interchanging F_1, F_2 yields the result. ■

3 $3 \times \ell$ Berge hypergraphs

This section provides an explcit classification of the asymptotic bounds $\mathrm{Bh}(m,F)$. Let

$$G_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Theorem 3.1 *Let F be a $3 \times \ell$ (0,1)-matrix.*
(Constant Cases) If $F \prec [I_3 \,|\, t \cdot \mathbf{0}_3]$, then $\mathrm{Bh}(m,F)$ *is* $\Theta(1)$.
(Linear Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_1$ or $\mathbf{1}_2$ and if $F \prec [G_1 \,|\, t \cdot [\mathbf{0} \,|\, I_3]] = H(1,3,t)$ then $\mathrm{Bh}(m,F) = \Theta(m)$.
(Quadratic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_2$ or G_2, or $\mathbf{1}_3$ and if $F \prec [\mathbf{1}_3 \,|\, t \cdot G_2] = H(2,3,t)$ for some t, then $\mathrm{Bh}(m,F) = \Theta(m^2)$.
(Cubic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_3$ then $\mathrm{Bh}(m,F) = \Theta(m^3)$.

Proof: The lower bounds follow from Lemma 2.5 and Lemma 2.6.

The constant upper bound for $[I_3 \,|\, t \cdot \mathbf{0}_3]$ is given by Theorem 2.7 combined with Lemma 2.21 to add columns of 0's. An exact linear bound for G_1 is in Theorem 3.2. The linear bound for $[G_1 t \cdot [\mathbf{0} \,|\, I_3]] = H(1,3,t)$ and the quadratic upper bound for $[\mathbf{1}_3 \,|\, t \cdot G_2] = H(2,3,t)$ follow from Theorem 2.15. The cubic upper bound for $t \cdot K_3$ follows from Theorem 2.1.

To verify that all 3-rowed matrices are handled we first note that $\mathrm{Bh}(m, 2 \cdot \mathbf{1}_3)$ is $\Theta(m^3)$. Consider matrices F with $2 \cdot \mathbf{1}_3 \not\prec F$. Then $F \prec H(2,3,t)$ and so $\mathrm{Bh}(m,F)$

is $O(m^2)$. If $2\cdot\mathbf{1}_2$, $\mathbf{1}_3$ or $G_2 \prec F$ then $\mathrm{Bh}(m,F)$ is $\Omega(m^2)$. Now assume $2\cdot\mathbf{1}_2$, $\mathbf{1}_3$ and $G_2 \not\prec F$. Then $G(F)$ (from Definition 2.4) has no 3-cycle nor a repeated edge and so $F \prec H(1,3,t)$. Then $\mathrm{Bh}(m,F)$ is $O(m)$. If $2\cdot\mathbf{1}_1$ or $\mathbf{1}_2 \prec F$ then $\mathrm{Bh}(m,F)$ is $\Omega(m)$. The only 3-rowed F with $2\cdot\mathbf{1}_1 \not\prec F$ and $\mathbf{1}_2 \not\prec F$ satisfies $F \prec [I_3\,|\,t\cdot\mathbf{0}_3]$. ■

The following theorem is an example of the difference between Berge hypergraphs and configurations. Note that $\mathrm{forb}(m,G_1)=2m$ [7].

Theorem 3.2 $\mathrm{Bh}(m,G_1)=\lfloor\frac{3}{2}m\rfloor+1$

Proof: Let $A\in \mathrm{BAvoid}(m,F)$. Then A has at most $m+1$ columns of sum 0 or 1. Consider two columns of A of column sum at least 2. If there is a row that has 1's in both column i and column j then we find a Berge hypergraph G_1. Thus columns of column sum at least 2 must occupy disjoint sets of rows and so there are at most $\lfloor\frac{m}{2}\rfloor$ columns of column sum at least 2. This yields the bound. Then we can form an $A\in\mathrm{BAvoid}(m,F)$ with $\|A\|=\lfloor\frac{3}{2}m\rfloor+1$. ■

4 $4\times\ell$ Berge hypergraphs

Given a (0,1)-matrix F, we denote by $r(F)$ (the reduction of F) the submatrix obtained by deleting all columns of column sum 0 or 1. In view of Lemma 2.20, we have that $\mathrm{Bh}(m,F)$ is $O(\mathrm{Bh}(m,r(F)))$. On 4 rows, there is an interesting and perhaps unexpected result.

Theorem 4.1 *[5]* $\mathrm{forb}(m,\{I_2\times I_2, T_2\times T_2\})$ *is* $\Theta(m^{3/2})$ *where*

$$I_2\times I_2 = C_4 = \begin{bmatrix} 1&1&0&0\\ 0&0&1&1\\ 1&0&1&0\\ 0&1&0&1 \end{bmatrix},\quad T_2\times T_2 = \begin{bmatrix} 1&1&1&1\\ 0&0&1&1\\ 1&1&1&1\\ 0&1&0&1 \end{bmatrix}.$$

The above result uses the lower bound construction (projective planes) from the much cited paper of Kővari, Sós and Turán.

Theorem 4.2 *[17]* $f(C_4, I_{m/2}\times I_{m/2})$ *is* $\Theta(m^{3/2})$.

We conclude a Berge hypergraph result much in the spirit of Gerbner and Palmer [15]. They maximized a different extremal function: essentially the number of 1's in a matrix in $\mathrm{BAvoid}(m,C_4)$.

Theorem 4.3 $\mathrm{Bh}(m,C_4)$ *is* $\Theta(m^{3/2})$

Proof: The lower bound follows from [17]. It is straightforward to see that $C_4 \prec T_2 \times T_2$ and then we apply Theorem 4.1 for the upper bound. ■

We give an alternative argument in Section 6 that handles $F = I_2 \times I_s$ for $s \geq 2$. Other 4-rowed Berge hypergraph cases are more straightforward. Let

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; H_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; H_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; H_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$H_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad H_6 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad H_7 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 4.4 *Let F be a $4 \times \ell$ (0,1)-matrix.*
(Constant Cases) If $F \prec [I_4 \,|\, t \cdot \mathbf{0}_4]$, then $\mathrm{Bh}(m, F)$ is $\Theta(1)$.
(Linear Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_1$ or $\mathbf{1}_2$ and if $r(F)$ is a configuration in H_1 or H_2 then $\mathrm{Bh}(m, F) = \Theta(m)$.
(Subquadratic Cases) If $r(F)$ is C_4, then $\mathrm{Bh}(m, F)$ is $\Theta(m^{3/2})$.
(Quadratic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_2$ or G_2, or $\mathbf{1}_3$ and if $F \prec H(2, 4, t)$ for some t, then $\mathrm{Bh}(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_3$ or $\mathbf{1}_4$ or K_4^2 or H_6 or H_7 and if $F \prec H(3, 4, t)$ then $\mathrm{Bh}(m, F) = \Theta(m^3)$.
(Quartic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_4$ then $\mathrm{Bh}(m, F) = \Theta(m^4)$.

Proof: The lower bounds follow from Lemma 2.5, Lemma 2.6 and Theorem 4.2.

The constant upper bound for $[I_4 \,|\, t \cdot \mathbf{0}_4]$ is given by Theorem 2.7 combined with Lemma 2.21 to add columns of 0's. The linear upper bound for F where $G(F)$ is a tree (or forest) follows from Theorem 6.5. There are only two trees on 4 vertices namely H_1 and H_2. Note $[H_2 \,|\, t \cdot [\mathbf{0}_4 \,|\, I_4]] = H(1, 4, t)$. Thus $\mathrm{Bh}(m, [H_2 \,|\, t \cdot [\mathbf{0}_4 \,|\, I_4]])$ is $O(m)$ by Theorem 2.15. Also $\mathrm{Bh}(m, [H_1 \,|\, t \cdot [\mathbf{0}_4 \,|\, I_4]])$ is $O(m)$ by Reduction Lemma 2.14. Now Theorem 4.3 establishes $\mathrm{Bh}(m, C_4)$. The quadratic upper bound for $H(2, 4, t)$ and the cubic upper bound for $H(3, 4, t)$ follow from Theorem 2.15. The quartic upper bound for $t \cdot K_4$ follows from Theorem 2.1.

To verify that all 4-rowed matrices are handled we first note that $\mathrm{Bh}(m, 2 \cdot \mathbf{1}_4)$ is $\Theta(m^4)$. Consider matrices F with $2 \cdot \mathbf{1}_4 \not\prec F$. Then $F \prec H(3, 4, t)$ and so $\mathrm{Bh}(m, F)$ is $O(m^3)$. If $2 \cdot \mathbf{1}_3 \prec F$, then $\mathrm{Bh}(m, F)$ is $\Omega(m^3)$ by Lemma 2.6. If $\mathbf{1}_4$, K_4^2, H_6 or $H_7 \prec F$ then $\omega(G(F)) = 4$ and so $\mathrm{Bh}(m, F)$ is $\Omega(m^3)$ by Lemma 2.5.

The column minimal simple (0,1)-matrices F with $\omega(G(F)) = 4$ and with column sums at least 2 are $\mathbf{1}_4$, K_4^2, H_5, H_6 and H_7. Since $H_6 \prec H_5$ it suffices to drop H_5 from the list. Now assume $\omega(G(F)) \leq 3$ and so $\mathbf{1}_4$, K_4^2, H_6 or $H_7 \not\prec F$. Also assume

$2\cdot\mathbf{1}_3 \not\prec F$. Let $3,4$ be the rows of F so that no column has 1's in both rows $3,4$. Then there are only two possible different columns of sum 3 in F and since $2\cdot\mathbf{1}_3 \not\prec F$, F has at most 2 (different) columns of sum 3. Hence $F \prec H(2,4,t)$ and $\mathrm{Bh}(m,F)$ is $O(m^2)$.

If $2\cdot\mathbf{1}_2 \prec F$, then $\mathrm{Bh}(m,F)$ is $\Omega(m^2)$ by Lemma 2.6. If $\mathbf{1}_3$ or $G_2 \prec F$, then $\mathrm{Bh}(m,F)$ is $\Omega(m^2)$ by Lemma 2.5. Now assume $F \prec H(2,4,t)$ but $2\cdot\mathbf{1}_2$, $\mathbf{1}_3$ and $G_2 \not\prec F$. Then $G(F)$ (from Definition 2.4) has no 3-cycle nor a repeated edge and so $G(F)$ is a subgraph of $K_{2,2}$ or $K_{1,3}$. In the latter case, $F \prec H(1,4,t)$. Then $\mathrm{Bh}(m,F)$ is $O(m)$. In the former case, $F \prec H((2,2),t)$ and so Theorem 4.3 applies to show that $\mathrm{Bh}(m,F)$ is $O(m^{3/2})$.

If $2\cdot\mathbf{1}_1$ or $\mathbf{1}_2 \prec F$ then $\mathrm{Bh}(m,F)$ is $\Omega(m)$. If $C_4 \prec F$, then $\mathrm{Bh}(m,F)$ is $\Omega(m^{3/2})$ by Theorem 4.3. The only subgraph of $K_{2,2}$ that contains C_4 and has no 3-cycle is C_4. The only 4-rowed F with $2\cdot\mathbf{1}_1 \not\prec F$, $\mathbf{1}_2 \not\prec F$ and $C_4 \not\prec F$ satisfies $F \prec [I_4\,|\,t\cdot\mathbf{0}_4]$. ■

We give some exact linear bounds.

$$\text{Let } H_8 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

For the following you may note that $\mathrm{forb}(m,H_8)$ is $\binom{m}{2}+2m-1$ and $\mathrm{forb}(m,H_2)$ is $\Theta(m^2)$ [7].

Theorem 4.5 *Assume* $m \geq 5$. *Then* $\mathrm{Bh}(m,H_8) = 2m$.

Proof: Let $A \in \mathrm{BAvoid}(m,H_8)$. Assume that A is a downset by Lemma 1.5. Let $A' = r(A)$. Since H_8 has column sums 2 then $\mathrm{Bh}(m,H_8) \leq \|A'\| + m + 1$. If A' has a column of column sum 4 (or more), then $H_8 \prec A'$ since H_8 has only 4 rows and is simple. If A' has a column of sum 3 say with 1's on rows 1,2,3, then we find $[K_3^3 K_3^2]$ in those 3 rows. Assume A' has a column of column sum 3, say with 1's in rows 1,2,3. Then if A' has either a column of sum 3 with at least one 1 in rows 1,2,3 and one 1 not in rows 1,2,3 or a column with at least 2 1's not in rows 1,2,3 then $H_8 \prec A'$ (using the fact that A is a downset). Thus if A' has a column of sum 3 with 1's in rows 1,2,3 then it has no columns with 1's not in rows 1,2,3 and so $\|A'\| = 4$. Thus for $m \geq 5$, $\|A'\| \leq m-1$ and so $\mathrm{Bh}(m,H_8) \leq 2m$.

If A' has only columns of sum 2 then we deduce that $\|A'\| \leq m-1$ and so $\mathrm{Bh}(m,H_8) \leq 2m$.

The construction to achieve the bound is to take the $m-1$ columns of sum 2 that have a 1 in row 1 as well as all columns of sum 0 or 1. We conclude that $\mathrm{Bh}(m,H_8) = 2m$. ■

Theorem 4.6 $\mathrm{Bh}(m,H_2) = 4\lfloor m/3 \rfloor + m + 1$.

Proof: Proceed as above. Let $A \in \mathrm{BAvoid}(m, H_2)$. Assume that A is a downset by Lemma 1.5. Let $A'=r(A)$ then $\mathrm{Bh}(m, H_2) \leq \|A'\| + m + 1$ since H_2 has column sums all 2. If A' has a column of column sum 4 (or more), then $H_2 \prec\!\!\prec A'$ since H_2 has only 4 rows and is simple. If A' has a column of sum 3, say with 1's on rows 1,2,3, then we find $[K_3^3 K_3^2]$ in those 3 rows. If A' has such a column of column sum 3, then A' cannot have a column with a 1 in row 1 and a 1 in row 4 else $F \prec\!\!\prec A'$ using the fact that A is a downset (using the columns with 1's in rows 1,2 and the column with 1's in rows 1,3 and the column with 1's in rows 1,4). Thus the number of columns of sum 3 is at most $\lfloor m/3 \rfloor$.

Let t be the number of columns of sum 3. If $m = 3t$, then we can include all columns of sum 2 that are in the downset of the columns of sum 3. All other columns of sum 2 have their 1's in the $m - 3t$ rows disjoint from those of the 1's in the columns of sum 3. The columns of sum 2, when interpreted as a graph, cannot have a vertex of degree 3 else $H_2 \prec\!\!\prec A$. So the number of columns of sum 2 is at most $m - 3t$ for $m - 3t \geq 3$ and 0 otherwise. This yields an upper bound.

A construction to achieve our bound is to simply take $\lfloor m/3 \rfloor$ columns of sum 3 each having their 1's on disjoint sets of rows and then, for each column of sum 3, add 3 columns of sum 2 whose 1's lie in the rows occupied by the 1's of the column of sum 3. ■

5 $5 \times \ell$ Berge hypergraphs

First we give the 5-rowed classification which requires Conjecture 7.1 to be true.

Theorem 5.1 *Let F be a $5 \times \ell$ (0,1)-matrix. Assume* $\mathrm{Bh}(m, \mathbf{1}_1 \times C_4)$ *is* $\Theta(m^2)$.
(Constant Cases) If $F \prec\!\!\prec [I_5 \,|\, t \cdot \mathbf{0}_5]$, then $\mathrm{Bh}(m, F)$ *is* $\Theta(1)$.
(Linear Cases) If F has a Berge hypergraph $\mathbf{1}_2$ or $[1\,1]$ and if $r(F)$ is a vertex-edge incidence matrix of a tree then $\mathrm{Bh}(m, F) = \Theta(m)$.
(Subquadratic Cases) If $r(F)$ is is a vertex-edge incidence matrix of a bipartite graph G with a cycle then $\mathrm{Bh}(m, F)$ *is* $\Theta(\mathrm{ex}(m, G))$ *i.e.* $\Theta(m^{3/2})$.
(Quadratic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_2$ or $\chi(G(F)) \geq 3$, and if $r(F)$ is a configuration in $H(2, 5, t)$ from (2) for some t or in $H((1, 2, 2), t)$ from (3) , then $\mathrm{Bh}(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_3$ or $\mathbf{1}_4$ or K_4^2 or H_6 or H_7 and if $F \prec\!\!\prec H(3, 5, t)$ from (2) for some t then $\mathrm{Bh}(m, F) = \Theta(m^3)$.
(Quartic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_4$ or if $\omega(G(F)) = 5$ and $F \prec\!\!\prec H(4, 5, t)$ then $\mathrm{Bh}(m, F) = \Theta(m^4)$.
(Quintic Cases) If F has a Berge hypergraph $2 \cdot \mathbf{1}_5$ then $\mathrm{Bh}(m, F) = \Theta(m^5)$.

Proof: The lower bounds follow from Lemma 2.5, Lemma 2.6 and also Theorem 4.2. Note that a bipartite graph on 5 vertices with a cycle must have a 4-cycle. In the quadratic cases, we could have listed three minimal examples of Berge hypergraphs with $\chi(G(F)) \geq 3$, namely $\mathbf{1}_3$, G_2 or the 5×5 vertex edge incidence matrix of the 5-cycle.

The constant upper bound for $[I_5 \,|\, t \cdot \mathbf{0}_5]$ is given by Theorem 2.7 combined with Lemma 2.21 to add columns of 0's. The linear upper bound for F where $G(F)$ is a tree (or forest) follows from Theorem 6.5. There are a number of trees on 5 vertices. Let F be the vertex-edge incidence matrix of a bipartite graph on 5 vertices that contains a cycle and hence contains C_4. Thus $F \prec I_2 \times I_3$ and so Theorem 6.1 establishes that $\mathrm{Bh}(m, F)$ is $O(m^{3/2})$. The quadratic upper bound for $H(2,5,t)$ and the cubic upper bound for $H(3,5,t)$ and the quartic upper bound for $H(4,5,t)$ follow from Theorem 2.15. The quadratic bound for $H((1,2,2),t)$ is Theorem 2.18 under the assumption $\mathrm{Bh}(m, \mathbf{1}_1 \times C_4)$ is $\Theta(m^2)$. The quintic upper bound for $t \cdot K_5$ follows from Theorem 2.1.

To verify that all 5-rowed matrices are handled we first note that $\mathrm{Bh}(m, 2 \cdot \mathbf{1}_5)$ is $\Theta(m^5)$. Consider matrices F with $2 \cdot \mathbf{1}_5 \not\prec F$. Then $F \prec H(4,5,t)$ and so $\mathrm{Bh}(m, F)$ is $O(m^4)$.

If $2 \cdot \mathbf{1}_4 \prec F$, then $\mathrm{Bh}(m, F)$ is $\Omega(m^4)$ by Lemma 2.6. If $\omega(G(F)) = 5$ then $\mathrm{Bh}(m, F)$ is $\Omega(m^4)$ by Lemma 2.5.

Now assume $\omega(G(F)) \leq 4$ and $2 \cdot \mathbf{1}_4 \not\prec F$. Let $4, 5$ be the rows so that no column has 1's in both rows $4, 5$. Three columns of sum 4 in F either force $\omega(G(F) = 5$ or we have a column of sum 4 repeated. So F has at most 2 (different) columns of sum 4 and so $F \prec H(3,5,t)$ for some t which yields that $\mathrm{Bh}(m, F)$ is $O(m^3)$.

If $2 \cdot \mathbf{1}_3 \prec F$, then $\mathrm{Bh}(m, F)$ is $\Omega(m^3)$ by Lemma 2.6. If $\mathbf{1}_4$ or K_4^2 or H_6 or $H_7 \prec F$, then $\omega(G(F)) \geq 4$ and then $\mathrm{Bh}(m, F)$ is $\Omega(m^3)$ by Lemma 2.5.

Now assume $\omega(G(F)) \leq 3$ and $2 \cdot \mathbf{1}_3 \not\prec F$. If $\alpha(G(F)) \geq 3$, then by taking rows 3,4,5 to be the rows of an independent set of size 3, we have $F \prec H(2,5,t)$ and so $\mathrm{Bh}(m, F)$ is $O(m^2)$. The maximal graph on 5 vertices with $\omega(G(F)) \leq 3$ and $\alpha(G(F)) \leq 2$ is in fact $G(\mathbf{1}_1 \times C_4)$. Thus $F \prec H((1,2,2),t)$ for some t and by assumption $\mathrm{Bh}(m, F)$ is $O(m^2)$.

Now if $2 \cdot \mathbf{1}_2 \prec F$, then $\mathrm{Bh}(m, F)$ is $\Omega(m^2)$ by Lemma 2.6. If $\mathbf{1}_3$ or $K_3^2 \prec F$ then $\omega(G(F)) \geq 3$ and then $\mathrm{Bh}(m, F)$ is $\Omega(m^2)$ by Lemma 2.5. Now assume $\omega(G(F)) \leq 2$ and $2 \cdot \mathbf{1}_2 \not\prec F$. Thus the columns of F of sum at least 2 must have column sum 2 and there are no repeats of columns of sum 2. The graph $G(F)$ has no triangle. If it is not bipartite then $\chi(G(F)) \geq 3$ and then $\mathrm{Bh}(m, F)$ is $\Omega(m^2)$.

Now assume $2 \cdot \mathbf{1}_2 \not\prec F$ and $\chi(G(F)) \leq 2$ and so the columns of sum 2 of F form a bipartite graph $G(F)$ and there are no columns of larger sum. The graph $G(F)$ is either a tree in which case $\mathrm{Bh}(m, F)$ is $O(m)$ by Theorem 6.5 or if there is a cycle it must be C_4 and so $\mathrm{Bh}(m, F)$ is $\Omega(m^{3/2})$. But $G(F)$ is a subgraph of $K_{2,3}$ and so we may apply Theorem 6.1 (and Theorem 2.20) to obtain $\mathrm{Bh}(m, F)$ is $\Omega(m^{3/2})$.

If $2 \cdot \mathbf{1}$ or $\mathbf{1}_2 \prec F$ then $\mathrm{Bh}(m, F)$ is $\Omega(m)$. The only F with $2 \cdot \mathbf{1}_1 \not\prec F$ and $\mathbf{1}_2 \not\prec F$ satisfies $F \prec [I_5 \,|\, t \cdot \mathbf{0}_5]$ for some t. ∎

Attempting the classification for 6-rowed F would require bounds such as $\mathrm{Bh}(m, I_1 \times I_2 \times I_3)$ and $\mathrm{Bh}(m, I_2 \times I_2 \times I_2)$.

6 Berge hypergraphs from graphs

Let G be a graph and let F be the vertex-edge incidence graph so that $G(F) = G$. This section explores some connections of Berge hypergraphs F with extremal graph theory results. The first results provides a strong connection with $\mathrm{ex}(m, K_{s,t})$ and the related problem $\mathrm{ex}(m, T, H)$ (the maximum number of subgraphs T in an H-free graph on m vertices). Then we consider the case G is a tree (or forest).

Theorem 6.1 *Let $F = I_2 \times I_t$ be the vertex-edge incidence matrix of the complete bipartite graph $K_{2,t}$. Then $\mathrm{Bh}(m, F)$ is $\Theta(\mathrm{ex}(m, K_{2,t}))$ which is $\Theta(m^{3/2})$.*

Proof: It is immediate that $\mathrm{Bh}(m, F)$ is $\Omega(\mathrm{ex}(m, K_{2,t}))$ since the vertex-edge incidence matrix A of a graph on m vertices with no subgraph $K_{2,t}$ has $A \in \mathrm{BAvoid}(m, F)$.

Now consider $A \in \mathrm{BAvoid}(m, F)$. Applying Lemma 1.5, assume $T_i(A) = A$ for all i and so, when columns are viewed as sets, the columns form a downset. Thus for every column γ of A of column sum r, we have that there are all 2^r columns α in A with $\alpha \leq \gamma$. Assume for some column α of A of sum 2 that there are 2^{t-1} columns γ of A with $\alpha \leq \gamma$. But the resulting set of columns have the Berge hypergraph $\mathbf{1}_2 \times I_t$ by Theorem 2.7 and then, using the downset idea, will contain the Berge hypergraph F. Thus for a given column α of sum 2, there will be at most $2^{t-1} - 1$ columns γ of A with $\alpha < \gamma$. Thus $\|A\| \leq (2^{t-1})p$ where p is the number of columns of sum 2 in A. We have $p \leq \mathrm{ex}(m, K_{2,t})$ which proves the upper bound for $\mathrm{Bh}(m, F)$. ∎

Results of Alon and Shikhelman [1] are surprisingly helpful here. They prove very accurate bounds. For fixed graphs T and H, let $\mathrm{ex}(m, T, H)$ denote the maximum number of subgraphs T in an H-free graph on m vertices. Thus $\mathrm{ex}(m, K_2, H) = \mathrm{ex}(m, H)$. The following is their Lemma 4.4. The lower bound for $s = 3$ can actually be obtained from the construction of Brown [9]. The lower bounds for larger s have also been obtained by Kostochka, Mubayi and Verstraëtte [16].

Lemma 6.2 *[1] For any fixed $s \geq 2$ and $t \geq (s-1)! + 1$, $\mathrm{ex}(m, K_3, K_{s,t})$ is $\Theta(m^{3-(3/s)})$.*

We can use this directly in analogy to Theorem 6.1.

Theorem 6.3 *$\mathrm{Bh}(m, I_3 \times I_t)$ is $\Theta(m^2)$.*

Proof: Let $A \in \mathrm{BAvoid}(m, I_3 \times I_t)$. Applying Lemma 1.5, assume $T_i(A) = A$ for all i and so, when columns are viewed as sets, the columns form a downset. Thus for every column γ of A of column sum r, we have that there are all 2^r columns α in A with $\alpha \leq \gamma$. Let G be the graph associated with the columns of sum 2 and so a column of sum r corresponds to K_r in G. In particular the number of columns of sum 3 is bounded by $\mathrm{ex}(m, K_3, K_{3,t})$ since each column of sum 3 yields a triangle K_3. Assume for some column α of A of sum 3 that there are 2^{t-1} columns γ of A with $\alpha \leq \gamma$. But the resulting set of columns have the Berge hypergraph $\mathbf{1}_3 \times I_t$ by Theorem 2.7 and then, using the downset idea, will contain the Berge hypergraph $I_3 \times I_t$. Thus for a given column α of sum 3, there will be at most $2^{t-1}-1$ columns γ of A with $\alpha < \gamma$. Thus $\|A\| \le (2^{t-1})p + |E(G)|$ where p is the number of columns of sum 3 in A. We have $p \le \mathrm{ex}(m, K_3, K_{3,t})$. This yields $\|A\| \le 2^{t-1}\mathrm{ex}(m, K_3, K_{3,t}) + \mathrm{ex}(m, K_{3,t})$. Now the standard inequalities yield $\mathrm{ex}(m, K_{3,t})$ is $O(m^{5/3})$ and combined with Lemma 6.2 we obtain the upper bound. The lower bound would follow from taking construction of $\Theta(m^{3-(3/t)})$ columns of sum 3 from the triangles K_3 in Lemma 6.2. ∎

We could follow the above proof technique and verify, for example, that

$$\mathrm{Bh}(m, I_4 \times I_7) \text{ is } O\left(\mathrm{ex}(m, K_{4,7}) + \mathrm{ex}(m, K_3, K_{4,7}) + \mathrm{ex}(m, K_4, K_{4,7})\right)$$

using the idea that we can restrict our attention, for an asymptotic bound, to columns of sum 2,3,4. Note that Lemma 6.2 yields $\mathrm{ex}(m, K_3, K_{4,7})$ is $\Theta(m^{2+(1/4)})$ and so $\mathrm{Bh}(m, I_4 \times I_7)$ is $\Omega(m^{2+(1/4)})$. Thus $I_{m/2} \times I_{m/2}$ won't be the source of the construction. The paper [6] has some lower bounds (Lemma 4.3 in [6]):

Lemma 6.4 *[6] For any fixed r, $s \ge 2r-2$ and $t \ge (s-1)!+1$. Then*

$$\mathrm{ex}(m, K_r, K_{s,t}) \ge \left(\frac{1}{r!} + o(1)\right) m^{r-\frac{r(r-1)}{2s}}. \qquad ∎$$

Thus for some choices r, s, t, $\mathrm{ex}(m, K_r, K_{s,t})$ grows something like $\Omega(m^{r-\epsilon})$ which shows we can take many columns of sum r and still avoid $K_{s,t}$, i.e. $\mathrm{Bh}(m, K_{s,t})$ grows very large.

Theorem 6.5 *Let F be the vertex-edge incidence $k \times (k-1)$ matrix of a tree (or forest) T on k vertices. Then* $\mathrm{Bh}(m, F)$ *is* $\Theta(m)$.

Proof: We generalize the result for trees/forests in graphs. It is known that if a graph G has all vertices of degree $k-1$, then G contains any tree/forest on k vertices as a subgraph. We follow that argument but need to adapt the ideas to Berge hypergraphs. Let $A \in \mathrm{BAvoid}(m, F)$ with A being a downset. We will show that $\|A\| \le 2^{k-1}m$.

If A has all rows sums at least $2^{k-1}+1$ then we can establish the result as follows. If we consider the submatrix A_r formed by those columns with a 1 in row r, then I_{k-1} is a Berge hypergraph contained in the rows $[m]\backslash r$ of A_r (by Theorem 2.7). Thus the vertex corresponding to row r in $G(A)$ has degree at least $k-1$. Then $G(A)$ has a copy of the tree/forest T and since A is a downset, $F \prec A$, a contradiction.

If A has some rows of sum at most 2^{k-1}, then we use induction on m. Assume row r of A has row sum $t \le 2^{k-1}$. Then we may delete that row and the t columns with 1's in row r and the resulting $(m-1)$-rowed matrix A' is simple with $\|A\| = \|A'\| + t$. By induction $\|A'\| \le 2^{k-1}(m-1)$ and this yields $\|A\| \le 2^{k-1}m$. ∎

The following results shows a large gap between Berge hypergraph results and forbidden configurations results.

The following matrices will be used in our arguments.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_9 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_{10} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Lemma 6.6 *For $k \geq 5$, $\mathrm{forb}(m, H_1 \times \mathbf{0}_{k-4})$ is $\Theta(m^{k-3})$.*

Proof: The survey [7] has the result $\mathrm{forb}(m, F_7)$ is $\Theta(m^2)$ listed in the results on 5-rowed configurations F. We have $H_1 \times \mathbf{0}_1 \prec F_7$. Thus $\mathrm{forb}(m, H_1 \times \mathbf{0}_1)$ is $O(m^2)$. The upper bound for $k \geq 6$ follows by 'standard induction' in analogy to Lemma 2.11. We note that $H_1 \times \mathbf{0}_{k-4}$ has a $(k-2) \times l$ submatrix with K_2^0 on every pair of rows and so $\mathrm{forb}(m, H_1 \times \mathbf{0}_{k-4})$ is $\Omega(m^{k-3})$ by [3]. ■

Lemma 6.7 $\mathrm{forb}(m, H_2 \times \mathbf{0}_{k-4})$ *is $\Theta(m^{k-2})$.*

Proof: Theorem 6.1 of [7] yields $\mathrm{forb}(m, H_2)$ is $\Theta(m^2)$ and so by 'standard induction' in analogy to Lemma 2.11 we have $\mathrm{forb}(m, H_2 \times \mathbf{0}_{k-4})$ is $O(m^{k-2})$. Now $H_2 \times \mathbf{0}_{k-4}$ has a $(k-1) \times l$ submatrix with K_2^0 on every pair of rows and so by [3], $H_2 \times \mathbf{0}_{k-4}$ is $\Omega(m^{k-2})$. ■

Lemma 6.8 $\mathrm{forb}(m, H_9 \times \mathbf{0}_{k-6})$ *is $\Theta(m^{k-1})$.*

Proof: $H_9 \times \mathbf{0}_{k-6}$ has K_2^0 on every pair of rows and so by [3], $H_9 \times \mathbf{0}_{k-6}$ is $\Theta(m^{k-1})$. ■

Theorem 6.9 *Assume $k \geq 5$ and let F be the $k \times l$ vertex-edge incidence matrix of a forest T. There are 3 cases covering all possible F:*

i. $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-3})$ if and only if $F \prec H_1 \times \mathbf{0}_{k-4}$.*

ii. $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-2})$ if and only if $F \not\prec H_1 \times \mathbf{0}_{k-4}$ and $H_9 \not\prec F$.*

iii. $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-1})$ if and only if $H_9 \prec F$.*

Proof:
Assume $k \geq 5$. The three cases cover all possible F. Note that $\mathrm{forb}(m, F)$ is $\Omega(m^{k-3})$ by [3] since a single edge in T produces a column which has $k-2$ rows with $\mathbf{0}_2$ on every pair of rows. Also, because F is simple, then $\mathrm{forb}(m, F)$ is $O(m^{k-1})$ [7].

We note that H_1 corresponds to a path of three edges and H_2 corresponds to a vertex of degree 3 (three edges incident with the same vertex) and H_9 corresponds to three vertex disjoint edges and H_{10} corresponds to a path of two edges and an additional edge vertex disjoint from the path.

Case i): forb(m,F) is $\Theta(m^{k-3})$ if and only if $F \prec H_1 \times \mathbf{0}_{k-4}$.

Assume forb(m,F) is $\Theta(m^{k-3})$. If T has at most 2 edges (F has only two columns) then $F \prec H_1 \times 0_{k-4}$. Using Lemma 6.7, we deduce $H_2 \times \mathbf{0}_{k-4} \not\prec F$. Thus if T has 3 edges while it does not have a path of three edges ($H_1 \times 0_{k-4} \not\prec F$) and no vertex of degree 3 ($H_2 \times \mathbf{0}_{k-4} \not\prec F$) then by a simple graph argument, T either consists of a path of two edges $(x,y),(y,z)$ and a vertex disjoint edge (u,v) and so $F = H_{10} \times \mathbf{0}_{k-5}$ (up to isomorphism) or three vertex disjoint edges so $H_9 \times \mathbf{0}_{k-6} \prec F$. In the case $F = H_{10} \times \mathbf{0}_{k-5}$, $\mathbf{1}_2 \not\prec F|_{\{2,3\}}$ and $\mathbf{0}_2 \not\prec F|_{\{1,4\}}$ and $I_2 \not\prec F|_{\{1,2\}}$. Now Theorem 2.3 yields Bh(m,F) is $O(m^{k-2})$. Considering that $\mathbf{0}_2 \prec F|_{\{i,j\}}$ for all pairs $2 \le i < j \le k$ we deduce that F is not a configuration of the $k-2$ fold product $I^c_{m/k-2} \times I^c_{m/k-2} \times \cdots \times I^c_{m/k-2}$ and so Bh(m,F) is $\Omega(m^{k-2})$, a contradiction. In the case $H_9 \times \mathbf{0}_{k-6} \prec F$, then Lemma 6.8, yields forb(m,F) is $\Theta(m^{k-1})$, a contradiction. There is no forest T with 4 or more edges which does not have a path of three edges, has no vertex of degree 3, no three edges with two incident and the other edge vertex disjoint from the first two ($H_{10} \times \mathbf{0}_{k-5} \not\prec F$) , and no three vertex disjoint edges ($H_9 \not\prec F$). We conclude that $F \prec H_1 \times \mathbf{0}_{k-4}$.

If $F \prec H_1 \times \mathbf{0}_{k-4}$ then forb(m,F) is $O(m^{k-3})$ by Lemma 6.6 and so $\Theta(m^{k-3})$ by our observation for any tree T. This concludes Case i).

Case ii): forb(m,F) is $\Theta(m^{k-2})$ if and only if $F \not\prec H_1 \times \mathbf{0}_{k-4}$ and $H_9 \not\prec F$.

Assume forb(m,F) is $\Theta(m^{k-2})$. Using Case i), we deduce that $F \not\prec H_1 \times \mathbf{0}_{k-4}$. We deduce that $H_9 \not\prec F$ by Lemma 6.8. We now consider a forest T that is not contained in a path of three edges ($F \not\prec H_1 \times \mathbf{0}_{k-4}$) and does not have three vertex disjoint edges ($H_9 \not\prec F$). Using the properties of the forest we will show that there is a pair of rows r_1, r_2 with $\mathbf{1}_2 \not\prec F|_{\{r_1,r_2\}}$ and there is a pair of rows s_1, s_2 with $\mathbf{0}_2 \not\prec F|_{\{s_1,s_2\}}$ and a pair of rows t_1, t_2 with $I_2 \not\prec F|_{\{t_1,t_2\}}$. Then Theorem 2.3 yields that forb(m,F) is $O(m^{k-2})$.

If a tree does not have two vertex disjoint edges then the tree is a star say with a root u and edges $(u,v_1),(u,v_2),\ldots(u,v_t)$. If T has three non-trivial components, then $H_9 \prec F$ so we may assume T has at most two non-trivial components. If T has two non-trivial components then no (non trivial) component has two vertex disjoint edges (else T has three vertex disjoint edges) and so each component is a star. Let the roots of the two stars be u_1, u_2 and let v_1 be joined to u_1. Then $\mathbf{1}_2 \not\prec F|_{\{u_1,u_2\}}$ and $\mathbf{0}_2 \not\prec F|_{\{u_1,u_2\}}$ and $I_2 \not\prec F|_{\{u_1,v_1\}}$.

Assume the forest T has only one non-trivial component. If T is a star with root u and edges to $v_1, v_2, \ldots, v_t$ with $t \ge 3$, then $\mathbf{1}_2 \not\prec F|_{\{v_1,v_2\}}$ and $\mathbf{0}_2 \not\prec F|_{\{u,v_1\}}$ and $I_2 \not\prec F|_{\{u,v_1\}}$. If T is not a star, then T has a path of at least 3 edges. T cannot have a path of 5 edges since then T has three vertex disjoint edges. Assume the longest path in T is x, u_1, u_2, y. Then every other edge is incident with either u_1 or U_1 and moreover, since $F \not\prec H_1 \times \mathbf{0}_{k-4}$, there is one such edge say (u_1, z). Then $\mathbf{1}_2 \not\prec F|_{\{x,z\}}$ and $\mathbf{0}_2 \not\prec F|_{\{u_1,u_2\}}$ and $I_2 \not\prec F|_{\{u_1,x\}}$.

Assume the longest path in T is x, u_1, y, u_2, z. To avoid creating 3 vertex disjoint edges then the only edges incident with y are (u_1, y) and (y, u_2). All other edges of T are incident with either u_1 or u_2. Then $\mathbf{1}_2 \not\prec F|_{\{x,z\}}$ and $\mathbf{0}_2 \not\prec F|_{\{u_1,u_2\}}$ and $I_2 \not\prec F|_{\{u_1,x\}}$. Thus in all possibilities Theorem 2.3 yields forb(m,F) is $\Theta(m^{k-2})$

If $F \not\prec H_1 \times \mathbf{0}_{k-4}$ and $H_9 \not\prec F$ then by Case i), forb(m,F) is $\Omega(m^{k-2})$. Now in all the forests above we have forb(m,F) is $O(m^{k-2})$. Hence forb(m,F) is $\Theta(m^{k-2})$ and this concludes Case ii).

Case iii): forb(m,F) is $\Theta(m^{k-1})$ if and only if $H_9 \prec F$.

Assume forb(m,F) is $\Theta(m^{k-1})$. Then by our observations in Case ii), we deduce $H_9 \prec F$.

If $H_9 \prec F$, then because F has column sums 2, $H_9 \times \mathbf{0}_{k-6} \prec F$ and so forb(m,F) is $\Omega(m^{k-1})$ by Lemma 6.8. By our general observations above, forb(m,F) is $O(m^{k-1})$ and so forb(m,F) is $\Theta(m^{k-1})$. This concludes Case iii). ∎

7 Conjecture and Problems

We have used the following conjecture in Theorem 5.1.

Conjecture 7.1 *Bh$(m, \mathbf{1}_1 \times C_4)$ is $\Theta(m^2)$.*

What are the equivalent difficult cases for larger number of rows? The above would yield Bh$(m, \mathbf{1}_2 \times C_4)$ is $\Theta(m^3)$ by Lemma 2.11 but we do not predict Bh$(m, \mathbf{1}_1 \times I_2 \times I_3)$. For $k = 6$, we believe that $F = I_2 \times I_2 \times I_2$ will be quite challenging given an old result of Erdős [10].

Theorem 7.2 *[10] $f(I_2 \times I_2 \times I_2, I_{m/3} \times I_{m/3} \times I_{m/3})$ is $O(m^{11/4})$ and $\Omega(m^{5/2})$.*

We might predict that Bh$(m, I_2 \times I_2 \times I_2) = \Theta(f(I_2 \times I_2 \times I_2, I_{m/3} \times I_{m/3} \times I_{m/3}))$. and so Bh$(m, I_2 \times I_2 \times I_2)$ is between quadratic and cubic. Unfortunately we offer no improvement to the bounds of Erdős.

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