Forbidden Berge Hypergraphs

Richard Anstee,
Santiago Salazar
UBC, Vancouver

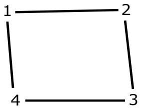
CanaDAM Ryerson University June 12, 2017

Introduction

Claude Berge and others created hypergraphs as a generalization of graphs. There are several hypergraph generalizations of paths and cycles. One generalization yields Berge paths and cycles. The definition of Berge Hypergraphs was given by Gerbner and Palmer (2015) and follows the same ideas. We consider the extremal set problem obtained by forbidding a single Berge Hypergraph

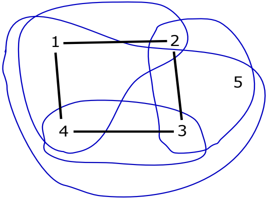
Santiago Salazar

Let F be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph H has F as a Berge Hypergraph and write $F \lll H$ if there are ℓ edges $E'_1, E'_2, \ldots, E'_\ell$ of H so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.



$$
\begin{aligned}
F &= C_4 \\
E_1 &= \{1, 2\} \\
E_2 &= \{2, 3\} \\
E_3 &= \{3, 4\} \\
E_4 &= \{1, 4\}
\end{aligned}
$$

Let F be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph H has F as a Berge Hypergraph and write $F \lll H$ if there are ℓ edges $E'_1, E'_2, \ldots, E'_\ell$ of H so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.



$$
\begin{array}{rcl}
F = C_4 & \lll & H \\
E_1 = \{1,2\} & & E'_1 = \{1,2,4\} \\
E_2 = \{2,3\} & & E'_2 = \{2,3,5\} \\
E_3 = \{3,4\} & & E'_3 = \{3,4\} \\
E_4 = \{1,4\} & & E'_4 = \{1,3,4,5\}
\end{array}
$$

We typically give our results using matrices. Define a matrix to be simple if it is a (0,1)-matrix with no repeated columns. A $k \times \ell$ (0,1)-matrix corresponds to a hypergraph (or set system) of ℓ edges on a ground set of k vertices where each column is viewed as the incidence matrix of an edge.

$$F = \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}\right] \end{array} \ll H = \begin{array}{c} \begin{matrix} E_1' & E_3' & E_4' & E_2' & \cdots \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 1 & \\ 0 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \end{matrix}\right] \end{array}$$

Patterns, Configurations and Berge Hypergraphs

Consider a (0,1)-matrix F. We say that A has F as a Berge Hypergraph if there is a submatrix B of A and a row and column permutation G of F so that $G \leq B$.

row/column order doesn't matter, 0's don't matter.

Patterns, Configurations and Berge Hypergraphs

Consider a (0,1)-matrix F. We say that A has F as a Berge Hypergraph if there is a submatrix B of A and a row and column permutation G of F so that $G \leq B$.

row/column order doesn't matter, 0's don't matter.

We say that A has F as a Pattern if there is a submatrix B of A so that $F \leq B$.

row/column order matters, 0's don't matter.

Patterns, Configurations and Berge Hypergraphs

Consider a (0,1)-matrix F. We say that A has F as a Berge Hypergraph if there is a submatrix B of A and a row and column permutation G of F so that $G \leq B$.

row/column order doesn't matter, 0's don't matter.

We say that A has F as a Pattern if there is a submatrix B of A so that $F \leq B$.

row/column order matters, 0's don't matter.

We say that A has F as a Configuration if there is a submatrix B of A and a row and column permutation G of F so that $G = B$.

row/column order doesn't matter, 0's matter.

Our Extremal Problem

Define $\|A\|$ as the number of columns of A. Define our extremal problem as follows:

$$\text{Avoid}(m, F) = \{A : A \text{ is } m\text{-rowed, simple, } F \not\prec A\},$$

$$\text{Bh}(m, F) = \max_A\{\|A\| : A \in \text{Avoid}(m, F)\}.$$

Forbidden Berge Hypergraph I_k

Theorem $\mathrm{Bh}(m, I_k) = 2^{k-1}$

The fact that this is a constant would follow from a result of Balogh and Bollobás (2005). This exact bound follows by induction or by the shifting argument given later.

Extremal Graph Theory

$\mathrm{ex}(m, G)$ is the maximum number of edges in a graph on m vertices which has no subgraph G.

Given a $k \times \ell$ (0,1)-matrix F, we can form a graph $G(F)$ on k vertices where we join i, j if there is a column in F with 1's in rows i, j. Alternatively replace the hyperedges in the hypergraph associated with F, by the cliques associated with each hyperedge and take the union of the edges.

Theorem $\mathrm{Bh}(m, F) \geq \mathrm{ex}(m, G(F)) + m + 1$

e.g.

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ has } G(F) = C_4.$$

Since $\mathrm{ex}(m, C_4) = \Theta(m^{3/2})$ then $\mathrm{Bh}(m, F)$ is $\Omega(m^{3/2})$.

$$C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Theorem (A., Koch, Raggi, Sali '14) $\mathrm{forb}(m, \{C_4, T_4\})$ is $\Theta(m^{3/2})$.

Corollary $\mathrm{Bh}(m, C_4)$ is $\Theta(m^{3/2})$.

Proof: $C_4 \lll T_4$ so avoiding C_4 as a Berge hypergraph will forbid both C_4 and T_4 as configurations (as well as some other configurations).

Theorem If $A \in \mathrm{Avoid}(m, F)$, then there exists an $A' \in \mathrm{Avoid}(m, F)$ with $\|A\| = \|A'\|$ and the columns of A' form a downset: namely if α is a column of A' and $\beta \leq \alpha$, then β is a column of A'.

Proof: Apply a shifting argument, replacing 1's by 0's in A as long as no repeated columns are created. The result is A'.

Forbidden Berge Hypergraph $I_2 \times I_4$

Definition The product $I_2 \times I_4$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Definition The product $I_2 \times I_4$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Note that $G(I_2 \times I_4) = K_{2,4}$.

Forbidden Berge Hypergraph $I_2 \times I_4$

Assume $A \in \mathrm{Avoid}(m, I_2 \times I_4)$ with

$$A = \begin{array}{c} i \\ j \\ \\ \\ \\ \\ \end{array} \overbrace{\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & \vdots & & & & \end{array}}^{>2^{4-1}}$$

Recall that $\mathrm{Bh}(m, I_4) = 2^3$.

$I_2 \times I_4$

$$A = \begin{array}{c} i \\ j \\ \\ \\ \\ \\ \end{array} \overbrace{\begin{array}{c|cc|c|cc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}}^{>2^{4-1}}$$

$$\vdots$$

Thus $I_2 \times I_4 \prec\!\!\prec A$ using the idea that A is a downset.
Hence if $I_2 \times I_4 \not\prec\!\!\prec A$ then for each pair of rows i, j, the number of columns of A with 1's on both rows i, j is at most 2^3.
Then the number of columns with three or more 1's is asymptotic to the number of columns of sum 2

Definition $\mathrm{ex}(m, K_\ell, G)$ is the maximum number of copies of K_ℓ in an m-vertex G-free graph.
Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

Definition $\mathrm{ex}(m, K_\ell, G)$ is the maximum number of copies of K_ℓ in an m-vertex G-free graph.
Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

Theorem (Alon, Shikhelman '15, Kostachka, et al '15)
Let s, t be given with $t \geq (s-1)! + 1$. Then $\mathrm{ex}(m, K_3, K_{s,t})$ is $\Theta(m^{3-(3/s)})$.

Theorem (Alon, Shikhelman '15, Kostachka, et al '15)
Let r, s, t be given with $s \geq 2r-2$, $t \geq (s-1)! + 1$

$$\mathrm{ex}(m, K_r, K_{s,t}) \geq \left(\frac{1}{r!} + o(1)\right) m^{r - \frac{r(r-1)}{2s}}$$

Definition $\mathrm{ex}(m, K_\ell, G)$ is the maximum number of copies of K_ℓ in an m-vertex G-free graph.
Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

Definition $\mathrm{ex}(m, K_\ell, G)$ is the maximum number of copies of K_ℓ in an m-vertex G-free graph.
Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.
Lemma Given $A \in \mathrm{Avoid}(m, I_3 \times I_k)$, where A is a downset, the number of columns of column sum ℓ ($\ell \geq 3$) in A is at most $\mathrm{ex}(m, K_\ell, K_{3,k})$.

Theorem $\mathrm{Bh}(m, I_3 \times I_k) \leq 1 + m + \mathrm{ex}(m, K_{3,k}) + 2^{k-1}\mathrm{ex}(m, K_3, K_{3,k}))$

Definition $\mathrm{ex}(m, K_\ell, G)$ is the maximum number of copies of K_ℓ in an m-vertex G-free graph.

Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

Lemma Given $A \in \mathrm{Avoid}(m, I_3 \times I_k)$, where A is a downset, the number of columns of column sum ℓ ($\ell \geq 3$) in A is at most $\mathrm{ex}(m, K_\ell, K_{3,k})$.

Theorem $\mathrm{Bh}(m, I_3 \times I_k) \leq 1 + m + \mathrm{ex}(m, K_{3,k}) + 2^{k-1}\mathrm{ex}(m, K_3, K_{3,k}))$

Theorem $\mathrm{Bh}(m, I_4 \times I_k) \leq 1 + m + \mathrm{ex}(m, K_{4,k}) + \mathrm{ex}(m, K_3, K_{4,k})) + 2^{k-1}\mathrm{ex}(m, K_4, K_{4,k})$

Trees

Let T_k be a tree on k vertices. A well known result for trees is $\mathrm{ex}(m, T_k)$ is $\Theta(m)$.

Theorem Let T_k be a tree on k vertices and let F be the k-rowed vertex-edge incidence matrix of T_k so $G(F) = T_k$ and F has column sums 2. Then
$\mathrm{Bh}(m, F)$ is $\Theta(m)$.

Trees

Let T_k be a tree on k vertices. A well known result for trees is $\mathrm{ex}(m, T_k)$ is $\Theta(m)$.

Theorem Let T_k be a tree on k vertices and let F be the k-rowed vertex-edge incidence matrix of T_k so $G(F) = T_k$ and F has column sums 2. Then
$\mathrm{Bh}(m, F)$ is $\Theta(m)$.

The situation is different for configurations

Theorem Let T_k be a tree on k vertices and let F be the k-rowed vertex-edge incidence matrix of T_k. Then
$\mathrm{forb}(m, F)$ is $\Theta(m^{k-1})$ or $\Theta(m^{k-2})$ or $\Theta(m^{k-3})$ depending on T_k.

Smallest Open Problem

$$C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ where } G(F) \text{ is } C_4.$$

$\mathrm{Bh}(m, C_4)$ is $\Theta(m^{3/2})$.

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem What is $\mathrm{Bh}(m, F)$?

We might guess that $\mathrm{Bh}(m, F)$ is $\Theta(m^2)$.

THANKS to those who have kept contributing to the CanaDAM series of conference. Another great event!