Multicoloured Matrices

Richard Anstee
UBC Vancouver

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I have had the good fortune of working with a number of coauthors in this area: Farzin Barekat, Ruiyuan (Ronnie) Chen, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Christina Koch, Linyuan (Lincoln) Lu, Connor Meehan, U.S.R. Murty, Miguel Raggi, Lajos Ronyai, and Attila Sali.

Definition A matrix is called an *r-matrix* if the entries belong to $\{0, 1, \ldots, r-1\}$.

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Definition We define $\|A\|$ to be the number of columns in A.

i.e. if A is an m-rowed simple 2-matrix then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \left[\begin{array}{ccc|c|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\}$$

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When we extend to r-matrices one possible interpretation is a set of divisors of $p_1^{r-1} p_2^{r-1} \cdots p_m^{r-1}$ where $p_1, p_2, \ldots, p_m$ are distinct primes.

Definition Given a matrix F, we say that A has F as a *configuration* written $F \prec A$ if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad \prec \quad \begin{bmatrix} 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & \boxed{0 \;\; 0 \;\; 2} & & 1 & \boxed{2} \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 2 & \boxed{0 \;\; 2 \;\; 2} & & 0 & \boxed{0} \end{bmatrix} = A$$

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$$\le r^m$$

Fundamental Result

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible (0,1)-columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

The Great Divide

Theorem (Füredi, Sali 12) Let $\mathcal{F}$ be a finite family of r-matrices.
1. If for every pair $i, j \in \{0, 1, \ldots, r-1\}$ there is some $F \in \mathcal{F}$ with all entries in F restricted to i or j, then $\text{forb}(m, r, \mathcal{F})$ is a polynomial in m.
2. If for some pair $i, j \in \{0, 1, \ldots, r-1\}$ there no $F \in \mathcal{F}$ with all entries in F restricted to i or j, then $\text{forb}(m, r, \mathcal{F})$ is $\Omega(2^m)$.

Extensions to r-matrices

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible (0,1)-columns on k rows.

It is attractive to consider a family $\mathcal{F}$ of matrices which is symmetric under permutations of the symbols. Let S_r denote all permutations σ of the symbols and let $\sigma(A)$ denote the matrix obtained by replacing each entry a_{ij} of A by $\sigma(a_{ij})$. Define

$$\mathcal{S}ym(F) = \{\sigma(F) : \sigma \in S_r\}$$

Note that if F has two different entries $\{a, b\}$ then $\sigma(F)$ has two different entries $\{\sigma(a), \sigma(b)\}$

Theorem (Füredi, Sali 12) forb$(m, r, \mathcal{S}ym(K_k))$ is $\Theta(m^{(k-1)\binom{r}{2}})$.

Let σ be a permutation of $\{0, 1, \ldots, r-1\}$ with $\sigma(0) = 0$ and $\sigma(1) = 2$.

$$\sigma(K_2) = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad \prec \quad \begin{bmatrix} 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & \boxed{\color{red}{0} \quad \color{red}{0} \quad \color{red}{2}} & & & 1 & \boxed{\color{red}{2}} \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 2 & \boxed{\color{red}{0} \quad \color{red}{2} \quad \color{red}{2}} & & & 0 & \boxed{\color{red}{0}} \end{bmatrix}$$

A Product Construction

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

Given p simple r-matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple r-matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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Let F be a simple (0,1)-matrix with no constant row.
Let $t = \text{forb}(m/p, \{F, F^c\})$.

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Let $A_{m/p}(0,1)$ denote the $m/p \times t$ simple $(0,1)$-matrix with no configuration F or F^c.
Let $A_{m/p}(i,j) = \sigma(A_{m/p}(0,1))$, where $\sigma(0) = i$ and $\sigma(1) = j$, which has no configuration $\sigma(F)$ or $\sigma(F^c)$.

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$B = A_{m/p}(0,1) \times A_{m/p}(0,2) \times \cdots \times A_{m/p}(r-2, r-1)$.

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We form the $\binom{r}{2}$-fold product
$B = A_{m/p}(0,1) \times A_{m/p}(0,2) \times \cdots \times A_{m/p}(r-2, r-1)$.
We verify that $\sigma(F), \sigma(F^c) \nprec B$ (F is simple, no constant row).
Then $\text{forb}(m, r, \mathcal{S}ym(F))$ is $\Omega(t^p)$ i.e. $\Omega((\text{forb}(m, \{F, F^c\})^{\binom{r}{2}})$.

New Results

We consider 2-rowed (0,1)-matrices F. There are some cases where the bound is small.

Theorem Let $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\text{forb}(m, r, \mathcal{S}ym(F)) = r$.

Proof: The only possible columns are the r constant columns.■

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$$\text{Let } F_p = \left[\begin{matrix} 0 & \overbrace{0\,0 \cdots 0}^{p} & \overbrace{1\,1 \cdots 1}^{p} \\ 0 & 1\,1 \cdots 1 & 0\,0 \cdots 0 \end{matrix}\right]$$

Theorem (A., Ferguson, Sali 01) Let p be given. Then $\text{forb}(m, 2, F) = pm - p + 1$ i.e. $\text{forb}(m, 2, F)$ is $\Theta(m)$.

Theorem (A., Sali 15) Let p be given. Then $\text{forb}(m, r, \mathcal{S}ym(F_p))$ is $O(m^{\binom{r}{2}})$.

Let

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Theorem (A., Barekat, Sali 11) $\text{forb}(m, 2, H) = 4m$.
Note that H is a maximal case for 4-rowed configurations that yield a linear bound.
Theorem (A., Sali 15) $\text{forb}(m, r, \mathcal{S}ym(H))$ is $\Theta(m^{\binom{r}{2}})$.

Induction

Given c', there exists a c so that

$$c(m-1)^t + c'(m-1)^{t-1} \leq cm^t$$

Induction

Let A_m be an m-rowed simple r-matrix with $F \not\prec A$ for $F \in \mathcal{F}$.

$$A_m = \left[\begin{array}{cccc} 00\cdots0 & 11\cdots1 & 22\cdots2 & \cdots \\ B_0 & B_1 & B_2 & \cdots \end{array}\right]$$

Let A_{m-1} denote the simple r-matrix obtained from $[B_0 B_1 B_2 \cdots]$ and for pair (a,b), let $A_{m-1}(a,b)$ denote the columns that are common to both B_a and B_b.

$$\|A_m\| \le \|A_{m-1}\| + \sum_{a,b \in \{0,1,\ldots,r-1\}} \|A_{m-1}(a,b)\|$$

If we can show that $\|A_{m-1}(a,b)\| \le c'(m-1)^{k-1}$ for all m, then there exists a c so that $\|A_n\| \le cn^k$ for all n.
We have that $F \not\prec A_m$ for $F \in \mathcal{F}$ and so if $F \prec \left[\begin{array}{cc} aa\cdots a & bb\cdots b \\ F' & F' \end{array}\right]$, then for pair (a,b), $F' \not\prec A_{m-1}(a,b)$.

Induction

If there exists a c'' so that matrices $A_{m-d}(a,b)$ at depth d have
$\|A_{m-d}(a,b)\| \leq c''$,
then $\|A\|$ is $O(m^d)$.

Some typical configurations are $I_k(a,b)$ and $T_k(a,b)$:

$$I_4(a,b) = \begin{bmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{bmatrix}, \qquad T_4(a,b) = \begin{bmatrix} b & b & b & b \\ a & b & b & b \\ a & a & b & b \\ a & a & a & b \end{bmatrix}$$

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$$\mathcal{T}_k(r) = \bigcup_{a,b\in\{0,1,\ldots,r-1\}} I_k(a,b) \quad \cup \bigcup_{a,b\in\{0,1,\ldots,r-1\}} T_k(a,b)$$

An Unavoidable Forbidden Family

Theorem (Balogh and Bollobás 05) Let k be given. Then

$$\text{forb}(m, \{I_k, I_k^c, T_k\}) \leq 2^{2^k}$$

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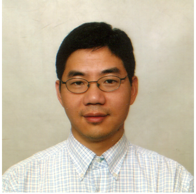
Theorem (A., Lu 14) Let k be given. Then there is a constant c

$$\text{forb}(m, r, \mathcal{T}_k(r)) \leq 2^{ck^{2r}}$$

The proof of the bound uses lots of induction and multicoloured Ramsey numbers: $R(k_1, k_2, \ldots, k_\ell)$ is the smallest value of n such than any colouring of the edges of K_n with ℓ colours $1, 2, \ldots, \ell$ will have some colour i and a clique of k_i vertices with all edges of colour i.

$$R(k_1, k_2, \ldots, k_\ell) \leq 2^{k_1+k_2+\cdots+k_\ell}$$



Linyuan (Lincoln) Lu

Another Divide

Theorem (A., Koch 13, A., Lu 14) Let $\mathcal{F}$ be a finite family of r-matrices. Let ℓ be the largest number of rows or columns in any $F \in \mathcal{F}$.

1. If for every $G \in \mathcal{T}_{2\ell}$ there is some $F \in \mathcal{F}$ with $F \prec G$, then $\text{forb}(m, r, \mathcal{F})$ is $O(1)$.
2. If for some $G \in \mathcal{T}_{2\ell}$ there is no $F \in \mathcal{F}$ with $F \prec G$, then $\text{forb}(m, r, \mathcal{F})$ is $\Omega(m)$.

$\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) =$

$$\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 2 & 2 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 2 & 2 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 2 \end{bmatrix},$$

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More asymptotically exact bounds

Theorem (A., Sali 15) Let $\mathcal{F}$ be a family of $(0,1)$-matrices. $\text{forb}(m, r, \mathcal{T}_k(r)\backslash\mathcal{T}_k(2) \cup \mathcal{F})$ is $\Theta(\text{forb}(m, \mathcal{F}))$.

Forbidding $\mathcal{T}_k(r)\backslash\mathcal{T}_k(2)$ is the same as restricting ourselves to $r = 2$, the case of (0,1)-matrices, at least asymptotically.

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Attila Sali

Thanks to Gary MacGillivray for organizing this conference!