Forbidden Configurations
A shattered history

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I have had the good fortune of working with a number of coauthors in this area: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Christina Koch, Linyuan (Lincoln) Lu, Connor Meehan, U.S.R. Murty, Miguel Raggi, Lajos Ronyai, and Attila Sali. A survey paper is available.

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i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Definition Given a matrix F, we say that A has F as a *configuration* written $F \prec A$ if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

Our Extremal Problem

Definition We define $\|A\|$ to be the number of columns in A.

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Example: $\text{forb}(m, \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]) = m + 1.$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

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Corollary *Let F be a $k \times \ell$ simple matrix. Then* $forb(m, F) = O(m^{k-1})$ $\quad (F \prec K_k)$

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VC-dimension gets used in Learning Theory and applied probability.

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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So $|sh(A)| = 7 \geq 6 = \|A\|$

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Theorem (Pajor 85) $\qquad |sh(A)| \geq \|A\|$.

Proof: Decompose A as follows:

$$A = \left[\begin{array}{cc} 0\,0\cdots 0 & 1\,1\cdots 1 \\ A_0 & A_1 \end{array} \right]$$

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$|sh(A)| \geq |sh(A_0)| + |sh(A_1)|$.
Hence $|sh(A)| \geq \|A\|$.

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

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Proof: Let $A \in \text{Avoid}(m, K_k)$.
Then $sh(A)$ can only contain sets of size $k-1$ or smaller.
Then

$$\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \geq |sh(A)| \geq \|A\|.$$

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration $F' \prec F$ such that

$$\text{forb}(m, F') = \text{forb}(m, F).$$

When $F' \prec F'' \prec F$, we deduce that

$$\text{forb}(m, F') = \text{forb}(m, F'') = \text{forb}(m, F).$$

Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the $(k+\ell) \times 1$ column of k 1's on top of ℓ 0's.
Let K_k^ℓ denote the $k \times \binom{k}{\ell}$ simple matrix of all columns of sum ℓ.

Miguel Raggi

Steven Karp

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\text{forb}(m, \mathbf{1}_4) = \text{forb}(m, K_4^3) = \text{forb}(m, K_4^2) = \text{forb}(m, K_4^1)$ $= \text{forb}(m, \mathbf{0}_4) = \text{forb}(m, 2 \cdot \mathbf{1}_3) = \text{forb}(m, 2 \cdot \mathbf{0}_3)$.

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The same is conjectured to be true for K_k for $k \geq 5$.

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
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Theorem (A., Meehan) For $m \geq 5$, we have
$\text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m, K_4)$.

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We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

Connor Meehan

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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Attila Sali

An Unavoidable Forbidden Family

Theorem (Balogh and Bollobás 05) Let k be given. Then

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Theorem (A., Lu 14) Let k be given. Then there is a constant c

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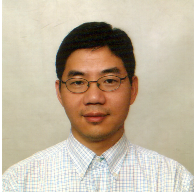
A construction taking all columns of column sum at most $k-1$ that arise from the $k-1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$ yields $\text{forb}(m, \{I_k, I_k^c, T_k\}) \geq \binom{2k-2}{k-1} \approx 2^{2k}$.
Probabalistic constructions of Balogh and Bollobás yield $\text{forb}(m, \{I_k, I_k^c, T_k\}) \geq c \cdot 2^{k \log k}$.

The proof uses lots of induction and multicoloured Ramsey numbers: $R(k_1, k_2, \ldots, k_\ell)$ is the smallest value of n such than any colouring of the edges of K_n with ℓ colours $1, 2, \ldots, \ell$ will have some colour i and a clique of k_i vertices with all edges of colour i. These numbers are readily bounded by multinomial coefficients:

$$R(k_1, k_2, \ldots, k_\ell) \le \binom{\sum_{i=1}^{\ell} k_i}{k_1\ k_2\ k_3 \cdots\ k_\ell}$$

$$R(k_1, k_2, \ldots, k_\ell) \le 2^{k_1+k_2+\cdots+k_\ell}$$



Linyuan (Lincoln) Lu

As part of our proof we wish to show that we cannot have a $u \times 2u$ large (0,1)-matrix of the form

0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
a	a	0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
b	b	c	c	0	1	$*$	$*$	$*$	$*$	$*$	$*$
d	d	e	e	f	f	0	1	$*$	$*$	$*$	$*$
g	g	h	h	i	i	j	j	0	1	$*$	$*$
k	k	l	l	m	m	n	n	o	o	0	1

One can interpret the entries of the matrix as 1×2 blocks yielding a $u \times u$ matrix with the blocks below the diagonal either $\boxed{0\ 0}$ or $\boxed{1\ 1}$ with blocks on the diagonal $\boxed{0\ 1}$ and arbitrary (0,1)-blocks above the diagonal.

We consider a colouring of the complete graph K_u with edge i, j getting a colour based on the entries in the block j, i and the block i, j.

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There are 6 colours to consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} * & 0 \\ 1 & 1 \end{bmatrix}$$

We are able to show that $u < R(k, k+1, k, k, k+1, k)$ and so we get a singly exponential bound on $u \leq 2^{6k+3}$. The proof has more to do than this but this is a critical step.

We say that the edge i, j is colour $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ if we have $0\,0$ in entry (j, i) and $1*$ in entry (i, j):

$$\begin{array}{ccccc} & i & & j \\ i & 0\,1 & & 1\,* \\ & & \ddots & \\ j & 0\,0 & & 0\,1 \end{array}$$

Now consider a clique of size $k+1$ of colour $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$:

$$\begin{array}{cccccccccc} 0 & 1 & 1 & * & 1 & * & 1 & * & 1 & * \\ 0 & 0 & 0 & 1 & 1 & * & 1 & * & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & * & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

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Design Theory

Let $s \cdot F$ denote $\overbrace{[F|F|\cdots|F]}^{s}$.

Theorem (A, Füredi 86) Let $s \geq 2$ be given. Then $\text{forb}(m, s \cdot K_k)$ is $\Theta(m^k)$.

Corollary (Füredi 83) Let F be a $k \times \ell$ (0,1)-matrix. Then $\text{forb}(m, F)$ is $\Theta(m^k)$.

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Theorem (A, Sali 14) Let α be given. $\text{forb}(m, m^{\alpha} \cdot K_k)$ is $\Theta(m^{k+\alpha})$

Note that we are having s grow with m. Our forbidden configuration is not fixed but depends on m.

Theorem $\mathrm{forb}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$.

Proof: We note that $[K_m^0 K_m^1 K_m^2 K_m^3] \in \mathrm{Avoid}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

Thus $\mathrm{forb}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) \geq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$.

(note that each pair of rows of has $(m-1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

We can argue, using the pigeonhole argument,

$$\mathrm{forb}(m, m \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{m-2}{3}\binom{m}{2}$$

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Thus forb$(m, m \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $\Theta(m^3)$.
Can we deduce the growth of forb$(m, m^\alpha \cdot \begin{bmatrix}1\\1\end{bmatrix})$?

Theorem

$$\text{forb}(m, (m + \binom{m-2}{2}) \cdot \begin{bmatrix}1\\1\end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \binom{m}{4}.$$

Note $[K_m^0 K_m^1 K_m^2 K_m^3 K_m^4] \in \text{Avoid}(m, (m + \binom{m-2}{2}) \cdot \begin{bmatrix}1\\1\end{bmatrix})$.

Definition Given integers $k \leq m$, let $\binom{[m]}{k}$ denote all k- subsets of $[m]$.

Definition Given parameters t, m, k, λ, a t-(m, k, λ) design $\mathcal{D}$ is a multiset of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are exactly λ blocks $B \in \mathcal{D}$ containing S.

Definition A t-(m, k, λ) design $\mathcal{D}$ is simple if $\mathcal{D}$ is a set (i.e. no repeated blocks).

If we have a t-(m, k, λ) simple design $\mathcal{D}$, then we can form a matrix M as the element-block incidence matrix associated with $\mathcal{D}$ and we deduce that

$$\|M\| = \left(\lambda\binom{m}{k}/\binom{k}{t}\right) \text{ and } M \in \text{Avoid}(m, (\lambda+1)\cdot\mathbf{1}_t)$$

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M has all columns of sum k. We can extend M:

$$A = [K_m^0\, K_m^1\, K_m^2 \cdots K_m^{k-1}\, M]$$

If we let $\mu = \binom{m-t}{0} + \binom{m-t}{1} + \cdots \binom{m-t}{k-1-t} + \lambda + 1$, then $A \in \text{Avoid}(m, \mu \cdot \mathbf{1}_t)$

We can deduce that

$$\text{forb}(m, \mu\cdot\mathbf{1}_t) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{k-1} + \lambda\binom{m}{k}/\binom{k}{t}$$

Breakthrough of Keevash on the Existence of Designs

Theorem (Keevash 14) Let $1/m \ll \theta \ll 1/k \leq 1/(t+1)$ and $\theta \ll 1$. Suppose that $\binom{k-i}{t-i}$ divides $\binom{m-i}{t-i}$ for $0 \leq i \leq r-1$. Then there exists a t-(m, k, λ) simple design for $\lambda \leq \theta m^{k-t}$.

Corollary (Weak Packing Idea) $\text{forb}(m, m^{\alpha} \cdot \mathbf{1}_k)$ is $\Theta(m^{k+\alpha})$.

For our purposes we don't care about equality but merely asymptotics. We use the Keevash result to establish lower bounds. His result is the first to establish this (of course his results do much more!).

Using the shifting idea, we have

$$\text{forb}(m, s \cdot K_k) = \text{forb}(m, s \cdot \mathbf{1}_k)$$

And this establishes the result:

Theorem (A., Sali 14) Let α be given. $\text{forb}(m, m^{\alpha} \cdot K_k)$ is $\Theta(m^{k+\alpha})$

Main Upper Bound Proof

Lemma Let F be a simple matrix and let $s > 1$ be given.
$\text{forb}(m, s \cdot F) \leq \sum_{i=1}^{m-1} (s-1) \cdot \text{forb}(m-i, F)$

Proof: We use the induction idea of A. and Lu 13. The idea is to temporarily allow the matrices to be non-simple in a restricted way.

$$\text{Let } F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We have $\text{forb}(m, F) = 4m$,

i.e. $\text{forb}(m, F)$ is $O(m)$.

Theorem Let $\alpha > 0$ be given. Using the Weak Packing idea, $\text{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{2+\alpha})$.

Proof:

$\text{forb}(m, m^\alpha \cdot F) \leq \sum_{i=1}^{m-1} m^\alpha \cdot \text{forb}(m-i, F) = m^\alpha \sum_{i=1}^{m-1} 4(m-i)$.

Now $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \prec F$ and so $m^\alpha \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \prec m^\alpha \cdot F$ from which we have

$\text{forb}(m, m^\alpha \cdot F) \geq \text{forb}(m, m^\alpha \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$. ■

An Open Problem

$$\text{Let} \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem (Frankl, Füredi, Pach 87) $\text{forb}(m, F) = \binom{m}{2} + 2m - 1$ i.e. $\text{forb}(m, F)$ is $O(m^2)$.

Theorem (A. and Lu 13) Let s be given. Then $\text{forb}(m, s \cdot F)$ is $\Theta(m^2)$.

Conjecture $\text{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{2+\alpha})$.

We can only prove that $\text{forb}(m, m^\alpha \cdot F)$ is $O(m^{3+\alpha})$.

Thanks to Ana Culibrk for the invite.